Inverse Boundary Value Problems in Electromagnetism

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Resumen

El objetivo principal de esta disertación es presentar unos resultados sobre determinación estable de los coeficientes de las ecuaciones de Maxwell a partir de mediciones no invasivas del campo electromagnético. Las mediciones que usamos como datos son de dos tipos: el primer tipo consiste en mediciones globales en la frontera de un dominio mientras que el segundo consiste en mediciones localizadas en una parte de dicha frontera. Desde el punto de vista de las aplicaciones la diferencia entre este tipo de mediciones se traduce en que en la primera situación tenemos acceso a toda la frontera del dominio, mientras que en la segunda situación la frontera tendrá una parte inaccesible sobre la que no podremos hacer mediciones.

Los resultados fundamentales de esta disertación consisten en un par de estimaciones, con módulo de continuidad logarítmico, en las que se relacionan las propiedades electromagnéticas del medio en el interior de un dominio con las mediciones en la frontera de dicho dominio. Posiblemente, este módulo de continuidad sea óptimo para el problema inverso que estudiamos.

Nuestros resultados se sostienen para el caso de dominios Lipschitz. Sin embargo, en el caso en el que sólo tenemos acceso a una parte de la frontera del dominio, necesitamos asumir ciertas restricciones geométricas sobre la parte inaccesible. En concreto, nuestros resultados para datos locales se sostienen sólo cuando la frontera del dominio es o bien parcialmente plana, o bien parcialmente esférica. A pesar de esta fuerte restricción son muchas las situaciones reales en las que la frontera presenta dicha forma.

Agradecimientos

Mi rostro no ha sido nunca mío, mis gestos
ya fueron realizados, mis palabras pronunciadas,
sentidos mi asombro, mi miedo, mi amargura.
No hay nada en mí que no haya ya pertenecido
a gentes que adivinaron mi existencia con la suya.

Iván Pérez Caro

Por agigantados e innovadores
que sean nuestros pasos,
hay que recordar que el escalón
que sustenta nuestros pies
es obra de otros. Por humildes o tímidas
que sean nuestras aportaciones,
no se debe olvidar que somos
reflejo de nuestros allegados. Por eso,
la gratitud y deuda contraída en nombres propios
sólo cabe en el libro del mundo.

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Chapter 1

Introduction

Justo antes de concluir la prueba el estudiante se volvió hacia su director y le preguntó:
–Si pudieras elegir un nombre para las cargas positivas y otro para las negativas, ¿cómo las llamarías?
–Pues a las cargas negativas las llamaría positivas y las positivas, por contraposición, negativas.
–Y eso, ¿por qué?
–Porque, antes que poeta, soy matemático.

In 1980 Calderón formulated an important inverse boundary value problem in electrostatic (see [9]). The problem –nowadays known as the Calderón problem– consists in determining the conductivity inside a body from non-invasive current and voltage measurements. This problem is the base of the so-called electrical impedance tomography. Possible applications of this imaging method include medical imaging, geophysical prospection and non-destructive testing of mechanical parts (see [6]).

The mathematical formulation of the Calderón problem is as follows. Let Ω be a bounded Lipschitz domain and let $\sigma \in L^\infty(\Omega)$ be a positive function describing the electric conductivity of the medium within Ω. In the absence of sources or sinks of electric current, and prescribed a potential $f \in H^{1/2}(\partial\Omega)$ on the boundary, the electrostatic potential u inside Ω solves the following Dirichlet problem

$$\left\{ \begin{array}{l} \nabla \cdot (\sigma \nabla u) = 0, \\ u|_{\partial\Omega} = f. \end{array} \right. \tag{1.1}$$

The uniqueness and the existence of $u \in H^1(\Omega)$ solving this problem are consequences of the Lax-Milgram lemma. The well-posedness of this problem allows us to define $\sigma \partial_N u|_{\partial\Omega}$ as an element of $H^{-1/2}(\partial\Omega)$, which describes the outgoing current. Thus, the current and voltage measurements on the boundary can be modeled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \longmapsto \sigma \partial_N u|_{\partial\Omega}.$$

The map $\Lambda_\sigma : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ is linear and bounded.

Mathematically, the Calderón problem consists in recovering the electric conductivity σ from the Dirichlet-to-Neumann map Λ_σ. In order to guarantee a unique and stable recovery of σ, it makes sense to answer the following questions:

- *Uniqueness*: Given $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ satisfying $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, can we ensure that $\sigma_1 = \sigma_2$?

- *Stability*: Does there exist a modulus of continuity b such that for any two $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ the following estimate holds

$$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leq b(\|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|)?$$

  Here $\|\cdot\|$ stands for the norm of operators from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

It is a natural question, not only from a mathematical point of view but for possible applications, the viability of setting the same kind of inverse boundary value problem in more general contexts like electromagnetism or in different physical frameworks like elasticity. Thus, in 1992 Somersalo, Isaacson and Cheney stated an inverse boundary value problem in electromagnetism inspired in the Calderón problem (see [40]). The goal of this problem is to determine the electric permittivity, the magnetic permeability and the electric conductivity in a unaccessible region from non-invasive measurements of the electromagnetic fields. The application areas include geophysical prospection, nondestructive testing and medical imaging. An important example is the detection of leukemia by using electromagnetic waves. This is possible because leukemia causes a representative change of the electric permittivity in the bone marrow. For more details see [11] and [12].

Before setting precisely the mathematical problem we recall the base of electrodynamics, that is, Maxwell's equations.

Maxwell's equations are a set of partial differential equations that describe the laws of electromagnetism in full generality. These equations are based on four laws:

- *Gauss's law for electric fields*: The flux of the *electric displacement* D through any closed surface $\partial\Omega$ is the free charge q enclosed within that surface.

- *Gauss's law for magnetic fields*: The flux of the *magnetic induction* B through any closed surface $\partial\Omega$ is zero.

- *Faraday's law*: A changing flux of the magnetic induction through a surface Γ induces a *motional electromotive force* –EMF for short– on the boundary of that surface. Additionally, a changing magnetic induction produces a circulating *electric field* E.

- *The Ampère-Maxwell law*: A free electric current I or a changing flux of the electric displacement through a surface Γ produces a circulating *magnetic field* H around the boundary of the surface.

When modeling mathematically these laws we can choose two different approaches of multivariable calculus, the vector field approach or the differential form approach. The most important advantage of the second one is its free coordinate character. This makes possible to write Maxwell's equations in the framework of differential manifolds. However, there is another advantage to choose the differential form approach instead of the vector field one. This is that differential forms recall very well the physical properties of the treated magnitudes. For example, the integration of 2-forms *restricted* to surfaces gives as result the flux, while the integration of 1-forms *restricted* to paths gives as result the circulation. Thus, it makes sense to write down the laws using differential forms:

- *Gauss's law for electric fields*:

$$\int_{\partial\Omega} D = \int_{\Omega} \rho.$$

- *Gauss's law for magnetic fields*:

$$\int_{\partial\Omega} B = 0.$$

- *Faraday's law*:

$$\mathrm{EMF} = -\frac{d}{dt}\int_{\Gamma} B, \qquad\qquad \int_{\partial\Gamma} E = -\int_{\Gamma} \partial_t B.$$

- *The Ampère-Maxwell law*:

$$\int_{\partial\Gamma} H = \int_{\Gamma} J + \frac{d}{dt}\int_{\Gamma} D.$$

Here D, B are 2-forms *restricted* to $\partial\Omega$ or Γ, ρ is the 3-form denoting the *volume free charge element* and $\partial\Omega$ is the surface enclosing Ω. Additionally E, H are 1-forms *restricted* to $\partial\Gamma$, J is the 2-form denoting the *sectional area current element* perpendicular to the direction of the current I, Γ is a surface and $\partial\Gamma$ stands for the path enclosing Γ. Finally, the four Maxwell's equations read

$$\begin{aligned} dD &= \rho, & -\partial_t D + dH &= J, \\ dB &= 0, & \partial_t B + dE &= 0. \end{aligned}$$

Maxwell's equations written as above explain how these observable physical quantities are related to each other. However, these equations are not enough to determine the quantities uniquely, we need a compatibility condition and structural assumptions on the medium. Thus, the compatibility condition is

$$\partial_t \rho = -dJ,$$

and the *structural* or *constitutive equations* described by Maxwell relate the magnitudes as follows:

$$D = \varepsilon * E, \quad B = \mu * H, \quad J = J_0 + \sigma * E.$$

Here ε is the *electric permittivity*, μ is the *magnetic permeability* and σ is the *electric conductivity*. The current density is divided in two parts, J_0 being the *forced current density* and $\sigma * E$ being the *ohmic current density* –driven by the electric field. Recall that $*$ is the Hodge star operator and that it relates 1-forms with 2-forms (see the beginning of Section 2.1). Roughly speaking, ε expresses the response of a dielectric to an applied electric field, and it quantifies the tendency of the material to form electric dipoles under the influence of an external electric field. Likewise, μ expresses the response of a material to an applied magnetic field. However, while electric dipoles induce an electric field that weakly opposes to the applied electric field, magnetic dipoles induce a magnetic field that may either oppose or reinforce the applied magnetic field. Finally, σ measures the ability of a material to conduct an electric current.

We remark that not all media obey the constitutive relations established above. Furthermore, not all media are isotropic. The medium is *isotropic* only if ε, μ, σ can be described by functions.

Somersalo *et al* proposed the problem in the domain of time-harmonic fixed frequency electromagnetism. This means that the time dependence of all fields is supposed to be $e^{-i\omega t}$ with $\omega > 0$ being the fixed frequency. Thus, the set of Maxwell's equations with the compatibility condition, the structural relations and $J_0 = 0$ becomes the *time-harmonic Maxwell equations*

$$\left\{ \begin{array}{l} dH + i\omega\gamma * E = 0 \\ dE - i\omega\mu * H = 0, \end{array} \right. \tag{1.2}$$

where γ denotes $\gamma = \varepsilon + i\sigma/\omega$. From this set of equations, we removed equations $d(\gamma * E) = 0$ and $d(\mu * H) = 0$ because they are redundant with the others. To check this, it is enough to take d in each equation of (1.2).

A detailed description of the problem is as follows. Let Ω be a bounded Lipschitz domain in the three-dimensional euclidean space. Assume the electromagnetic properties of the medium within Ω to be described by the non-negative functions $\varepsilon, \mu, \sigma \in L^\infty(\Omega)$. It is known that (1.2), complemented with a suitable prescribed data on the boundary, is well-posed. In fact, we have the following result.

Theorem 1 Let Ω be a bounded Lipschitz domain, $\omega \in \mathbb{C} \setminus \{0\}$ and $\mu, \varepsilon, \sigma \in L^\infty(\Omega)$ satisfying

$$\mu \geq \mu' > 0, \quad \varepsilon \geq \varepsilon' > 0, \quad \sigma \geq 0;$$

a. e. in Ω, with μ', ε' positive constants. Given $T \in TH(\partial\Omega)$, the problem of finding $E, H \in H(\Omega; \mathrm{curl})$ solving (1.2) in Ω and satisfying either $*(\nu \wedge E) = T$ or $*(\nu \wedge H) = T$ is well-posed for any $\omega \in \mathbb{C} \setminus \{0\}$ except for a subset of

$$\{\omega \in \mathbb{C} : -\left\|\sigma/\varepsilon\right\|_{L^\infty(\Omega)} \leq \operatorname{Im}\omega \leq 0\}$$

with no accumulation point in $\mathbb{C} \setminus \{0\}$.

Here ν stands for the 1-form defined by $\nu = e(N, \boldsymbol{.})$ with N the outward unit vector field normal to $\partial\Omega$, the boundary of Ω. The frequencies ω for which the direct problem is not well-posed are called *resonant frequencies*.

Theorem 1 allows us to model non-invasive measurements of electromagnetic fields on the boundary by means of the admittance or the impedance maps. The *admittance map* is defined as

$$\Lambda^{ad} : T \in TH(\partial\Omega) \longmapsto *(\nu \wedge H) \in TH(\partial\Omega),$$

where E, H is the solution for (1.2) with $*(\nu \wedge E) = T$. The *impedance map* is defined as

$$\Lambda^{im} : T \in TH(\partial\Omega) \longmapsto *(\nu \wedge E) \in TH(\partial\Omega),$$

where E, H is the solution for (1.2) with $*(\nu \wedge H) = T$. It is a consequence of Theorem 1 that these maps can only be defined out of resonant frequencies. Moreover, $\Lambda^{ad}, \Lambda^{im}$ are linear and bounded operators in $TH(\partial\Omega)$ whenever ω is non-resonant.

Mathematically, the problem proposed by Somersalo *et al* consists in recovering the electric permittivity ε, the magnetic permeability μ and the electric conductivity σ from the admittance map Λ^{ad}. In order to guarantee a unique and stable recovery of the parameters it makes sense to solve the following problems:

- *Uniqueness*: Consider $\mu_1, \gamma_1, \mu_2, \gamma_2 \in L^\infty(\Omega)$ and let $\Lambda_1^{ad}, \Lambda_2^{ad}$ denote their corresponding admittance maps for a non-resonant frequency. If we suppose $\Lambda_1^{ad} = \Lambda_2^{ad}$, can we ensure that $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$?

- *Stability*: Does there exist a modulus of continuity b such that for any two pairs $\mu_1, \gamma_1, \mu_2, \gamma_2 \in L^\infty(\Omega)$ the following estimate holds

$$\|\mu_1 - \mu_2\|_{L^\infty(\Omega)} + \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq b(\|\Lambda_1^{ad} - \Lambda_2^{ad}\|)?$$

  Here $\|\cdot\|$ stands for the norm of operators in $TH(\partial\Omega)$.

When trying to solve the stability problem from the admittance map, we have to face the problem of choosing $\omega > 0$ to be non-resonant for the class of coefficients to determine. How can we manage to solve this if the position of the resonant frequencies depends on the unknown coefficients? How can we know when our choice of frequency in the data is close to resonant frequencies?

In order to avoid this problem we shall model the boundary measurements by the Cauchy data set. Given a frequency $\omega > 0$, the *Cauchy data set* is defined as follows: $(T, S) \in C(\mu, \gamma)$ if and only if $(T, S) \in (TH(\partial\Omega))^2$ and there exists a pair $(E, H) \in (H(\Omega; \mathrm{curl}))^2$ solution of (1.2) satisfying $*(\nu \wedge E) = T$ and $*(\nu \wedge H) = S$.

When facing the problem of stability out of resonant frequencies one can use the operator norm to quantify the proximity of the boundary data. However, when we use the Cauchy data set to model our boundary measurements we can not profit anymore the operator norm to quantify this proximity. For this reason we need to introduce the following notion of distance between Cauchy data sets.

Definition 1.1 Let μ_1, γ_1 and μ_2, γ_2 be two pairs of coefficients. Consider ω a positive frequency and let C_j denote $C(\mu_j, \gamma_j)$. Let us define the pseudo-metric distance between the Cauchy data sets C_1 and C_2 as

$$\delta_C(C_1, C_2) = \max_{j \neq k} \sup_{\substack{(T_k, S_k) \in C_k \\ \|T_k\|_{TH(\partial\Omega))} = 1}} \inf_{(T_j, S_j) \in C_j} \|(T_j, S_j) - (T_k, S_k)\|_{(TH(\partial\Omega))^2}.$$

We say that δ_C is a pseudo-metric distance because if $\delta_C(C_1, C_2) = 0$, we can only ensure that $\overline{C_1} = \overline{C_2}$.

The definition of δ_C is inspired in the Hausdorff distance. Unlike the latter distance the former one is comparable to $\|\Lambda_1^{ad} - \Lambda_2^{ad}\|$ when ω is a non-resonant frequency for μ_j, γ_j with $j = 1, 2$. This means the following. Consider μ_j, γ_j with $j = 1, 2$, suppose that $\omega > 0$ is a non-resonant frequency for μ_j, γ_j and let Λ_j^{ad} its corresponding admittance map. Then, there exists a constant $C > 0$ such that

$$\delta(C_1, C_2) \leq \|\Lambda_1^{ad} - \Lambda_2^{ad}\| \leq C\delta(C_1, C_2),$$

where $C_j = \{(T_j, \Lambda^{ad} T_j) : T_j \in TH(\partial\Omega)\}$.

For practical purposes, the mathematical setting of this inverse problem might be useless because the data on the boundary are given by the tangential component of electric and magnetic fields. However, in practice it might be difficult to measure those tangential components on the boundary. Despite its apparent practical inviability, this problem is interesting because of its relation with Calderón's problem. Physically, the conductivity equation is obtained as a low-frequency limit of the time-harmonic Maxwell equations. In [25] Lassas proved that an appropriate restriction of the impedance mapping has a low-frequency limit, moreover, he gave a formula from which the Dirichlet-to-Neumann map for (1.1) can be calculated by using the low-frequency limit of the impedance map.

In practice, the right way of making non-invasive measurements of electromagnetic fields is by means of the far-field patter. That is to measure how plane waves are scattered by the medium where the parameters have to be determined. Thus, a more practical problem is the inverse scattering problem.

Assume U to be a smooth domain of the three-dimensional euclidean space and $\varepsilon(x) = \varepsilon_0$ and $\mu(x) = \mu_0$ for $x \notin U$. Consider the following plane waves

$$E_i(x) = e^{i\langle k,x\rangle} p, \qquad H_i(x) = e^{i\langle k,x\rangle} q.$$

Note that E_i, H_i satisfy Maxwell's equations in vacuum if

$$|k|^2 = \omega^2 \mu_0 \varepsilon_0, \quad \varepsilon_0 |p|^2 = \mu_0 |q|^2, \quad \langle p, q\rangle = 0.$$

Let E_s, H_s be solutions of

$$\begin{cases} d(H_i + H_s) + i\omega\gamma *(E_i + E_s) = 0 \\ d(E_i + E_s) - i\omega\mu *(H_i + H_s) = 0, \end{cases}$$

satisfying the (outgoing) *Silver-Müller radiation condition*, either

$$[*(\nu \wedge E_s) - *(\nu \wedge *(\nu \wedge H_s))]|_{|x|=\rho} = o\left(\frac{1}{\rho}\right)$$

or

$$[*(\nu \wedge H_s) + *(\nu \wedge *(\nu \wedge E_s))]|_{|x|=\rho} = o\left(\frac{1}{\rho}\right).$$

By using the representation of the scattered fields in terms of the Green functions, it is possible to derive an asymptotic representation of the fields

$$E_s(x) = E_\infty(\hat{x}; k; p) \frac{e^{i|k||x|}}{|x|} + o\left(\frac{1}{|x|}\right),$$
$$H_s(x) = H_\infty(\hat{x}; k; p) \frac{e^{i|k||x|}}{|x|} + o\left(\frac{1}{|x|}\right),$$

where $\hat{x} = x/|x|$ and the mutually orthogonal 1-forms E_∞ and H_∞ are the *electric* and *magnetic far-field patters* corresponding to the *polarization* p and *incident direction* k. Note that one only needs to specify one of these far-field patterns, the other can be obtained from the radiation condition.

The inverse scattering problem for fixed energy consists in recovering the material parameters γ and μ from the knowledge of the far-field pattern $E_\infty(\hat{x}; k; p)$ for all angles of observation $\hat{x} \in \mathbb{S}^2$, for all incident direction k at a single fixed frequency $|k|^2$ and three linearly independent polarizations p.

1.1 Stable determination from boundary data

In this section we give a partial answer for the question of stability of the inverse boundary value problem in electromagnetism. Assuming that our boundary data are given by Cauchy data sets, we use δ_C to give a stable determination of the electromagnetic parameters.

Our result requires certain stability of the problem on the boundary and since this has not been proven yet, we shall introduce some definitions.

Definition 1.2 Given two constants M, s such that $0 < M$, $0 < s < 1/2$, we shall say that the pair of coefficients μ, γ is *admissible* if they satisfy the following conditions.

(i) *Uniform ellipticity condition.* The coefficients $\gamma, \mu \in C^{1,1}(\overline{\Omega})$ satisfy

$$M^{-1} \le \operatorname{Re} \gamma(x) \qquad M^{-1} \le \mu(x);$$

for any $x \in \Omega$.

(ii) *A priori bound on the boundary.* The following a priori bound holds on the boundary

$$\|\gamma\|_{C^{0,1}(\partial\Omega)} + \|\mu\|_{C^{0,1}(\partial\Omega)} < M.$$

(iii) *A priori bound in the interior.* The following a priori bounds hold in the interior

$$\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \le M, \qquad \|\gamma\|_{H^{2+s}(\Omega)} + \|\mu\|_{H^{2+s}(\Omega)} \le M.$$

Definition 1.3 Let M, s be the constants given in Definition 1.2 and let ω be a positive frequency. We shall say that a pair μ, γ is in the *class of B-stable coefficients on the boundary at frequency* ω if μ, γ is an admissible pair and there exists a modulus of continuity B such that, for any other admissible pair $\tilde{\mu}, \tilde{\gamma}$, one has

$$\|\gamma - \tilde{\gamma}\|_{C^{0,1}(\partial\Omega)} + \|\mu - \tilde{\mu}\|_{C^{0,1}(\partial\Omega)} \le B\left(\delta_C(C, \tilde{C})\right),$$

$$\|\nabla(\gamma - \tilde{\gamma})\|_{L^\infty(\partial\Omega;\mathbb{C}^3)} + \|\nabla(\mu - \tilde{\mu})\|_{L^\infty(\partial\Omega;\mathbb{C}^3)} \le B\left(\delta_C(C, \tilde{C})\right).$$

Here $C, \tilde{C}$ are the Cauchy data sets associated to the pairs μ, γ and $\tilde{\mu}, \tilde{\gamma}$, respectively.

With these definitions at hand the stable determination of the electromagnetic coefficients can be stated as follows.

Theorem 2 Let Ω be a bounded Lipschitz domain and let ω be a positive frequency. Consider μ_1, γ_1 and μ_2, γ_2 any two pairs in the class of B-stable coefficients on the boundary at frequency ω, with B satisfying $|r| \leq B(|r|)$ for all $|r| < 1$. Then, there exists a constant $C = C(M)$ such that the following estimate holds

$$\|\gamma_1 - \gamma_2\|_{H^1(\Omega)} + \|\mu_1 - \mu_2\|_{H^1(\Omega)} \leq C|\log B(\delta_C(C_1, C_2))|^{-\lambda},$$

for some constant λ such that $0 < \lambda < 2/3s$. Here C_1, C_2 are the Cauchy data sets associated to the pairs μ_1, γ_1 and μ_2, γ_2, respectively.

As in the inverse conductivity problem, it should be possible to prove that any admissible pair is in the class of Hölder-stable coefficients on the boundary for any frequency ω, that is, with $B(|r|) = |r|^\alpha$ for $0 < \alpha < 1$. Notice that in the conductivity case a logarithmic module of continuity, as the one in Theorem 2, is optimal (see [28]).

For practical purposes the coefficients might be assumed to be constant or to be equal on the boundary, in those particular cases our result reads as follows.

Corollary 3 Let Ω be a bounded Lipschitz domain and let ω be a positive frequency. Consider μ_1, γ_1 and μ_2, γ_2 any two pairs of admissible coefficients. Assume that

$$\mu_1|_{\partial\Omega} = \mu_2|_{\partial\Omega},\ \partial_{x^j}\mu_1|_{\partial\Omega} = \partial_{x^j}\mu_2|_{\partial\Omega}, \quad \gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega},\ \partial_{x^j}\gamma_1|_{\partial\Omega} = \partial_{x^j}\gamma_2|_{\partial\Omega},$$

with $j = 1, 2, 3$. Then, there exists a constant $C = C(M)$ such that the following estimate holds

$$\|\gamma_1 - \gamma_2\|_{H^1(\Omega)} + \|\mu_1 - \mu_2\|_{H^1(\Omega)} \leq C|\log \delta_C(C_1, C_2)|^{-\lambda},$$

for some constant λ such that $0 < \lambda < 2/3s$.

The proof of Theorem 2 bases on an integral formula relating the boundary data with the coefficients in the interior by means of solutions for the time-harmonic Maxwell equations. In order to exploit the information coded in this formula, we need to construct special solutions, namely, exponential growing solutions. The construction of this kind of solutions requires to transform the time-harmonic Maxwell equations into a Schrödinger-type equation. On the latter equation we can perform Sylvester and Uhlmann's method to construct exponential growing solutions, afterwards we use these solutions to produce the wanted solutions for the original Maxwell's equations. The integral formula with these solutions yields information of a semi-linear elliptic system satisfied by the coefficients. Finally, we use a Carleman-type estimate to end up with the estimate established in Theorem 2.

1.2 Stable determination from local data

By the exigences of applications, it seems to be natural to answer the following question. Can we recover the properties of a medium from partial boundary data. It has been widely conjectured that only partial knowledge of the Dirichlet-to-Neumann map is required to recover the conductivity within Ω. However, this kind of problems are quite hard.

When setting an inverse boundary value problem with partial data, it is natural to assume that there is an unaccessible part of the boundary where measurements can not be taken. Thus, the problem becomes an inverse boundary value problem with local data, since our measurements are located in the accessible part of the boundary.

In the framework of electrostatic, the precise formulation of this problem is as follows. Let Γ be a non-empty proper open subset of $\partial\Omega$, the boundary of the domain Ω where we want to recover the conductivity. Set the Dirichlet-to-Neumann map localized to Γ as the map $\Lambda_\Gamma(\sigma) : H_0^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ defined by $\Lambda_\Gamma(\sigma)f = (\Lambda_\sigma f)|_\Gamma$ for $f \in H_0^{1/2}(\Gamma)$. Then the question is: Can we recover σ from the knowledge of the Dirichlet-to-Neumann map localized to Γ? As in the case where we have global boundary data, it makes sense to answer the following questions:

- *Uniqueness*: Given $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ satisfying $\Lambda_\Gamma(\sigma_1) = \Lambda_\Gamma(\sigma_2)$, can we ensure that $\sigma_1 = \sigma_2$?

- *Stability*: Does there exist a modulus of continuity b such that for any two $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ the following estimate holds

  $$\|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \leq b(\|\Lambda_\Gamma(\sigma_1) - \Lambda_\Gamma(\sigma_2)\|)?$$

  Here $\|.\|$ stands for the norm of operators from $H_0^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.

The space $H_0^{1/2}(\Gamma)$ is the space of $f \in H^{1/2}(\partial\Omega)$ such that $\operatorname{supp} f \subset \Gamma$. On the other hand, the space $H^{-1/2}(\Gamma)$ is the dual space of $H_0^{1/2}(\Gamma)$.

As far as the author knows, the only uniqueness result on the inverse conductivity problem with local data is due to Isakov and it reads as follows.

Theorem 1.1 (Isakov [19]) Let U be either a *suitable partially flat domain* or a *suitable partially spherical domain.* Consider $\sigma_1, \sigma_2 \in C^{1,1}(\overline{U})$ such that $\partial_N \sigma_1 = \partial_N \sigma_2 = 0$ on $\partial U \setminus \overline{\Gamma}$. If $\Lambda_\Gamma(\sigma_1) = \Lambda_\Gamma(\sigma_2)$, then $\sigma_1 = \sigma_2$.

The exact meaning of *suitable* is explained in Section 4.1.

The first idea in the Isakov's argument is to construct exponential growing solutions vanishing on $\partial U \setminus \overline{\Gamma}$, the inaccessible part of the boundary. In order to achieve this task, Isakov proposed a reflection argument which allows to construct solutions for the conductivity equation with the desired behavior on the boundary. In order to carry out the Isakov's approach it seems to be necessary to assume some geometrical restrictions about the domain, namely, the inaccessible part is supposed to be either part of a plane or part of a sphere. Moreover, he needs to keep the smoothness of σ_1 and σ_2 when performing the reflection argument, so he has to assume that $\partial_N \sigma_1 = \partial_N \sigma_2 = 0$ on $\partial U \setminus \overline{\Gamma}$. However, this last assumption seems to be unnecessary. Let us point out this.

When the part of the boundary of U is part of a sphere, we can use the inversion through an sphere to transform the spherical part in part of a plane. Thus, we can assume U to be included in $\mathbb{R}^3_- := \{x \in \mathbb{R}^3 : x^3 < 0\}$ with the flat part of ∂U laying on $\{x \in \mathbb{R}^3 : x^3 = 0\}$. Extend σ_j to $\overline{\mathbb{R}^3_-}$ keeping its smoothness and such that $\sigma(x) = 1$ for all $x \in \{x \in \overline{\mathbb{R}^3_-} : |x| > \rho\}$. So $\sigma_j \in C^{1,1}(\overline{\mathbb{R}^3_-})$. Define γ_j as $\gamma_j(x) = \sigma_j(x)$ for $x \in \overline{\mathbb{R}^3_-}$ and $\gamma_j(x) = \sigma_j(\mathcal{R}(x))$ for $x \in \mathbb{R}^3_+$, where $\mathcal{R}(x^1, x^2, x^3) = (x^1, x^2, -x^3)$. Note that $\gamma_j \in C^{0,1}(\mathbb{R}^3)$.

Lemma 1.1 Let $v_j \in H^1(\mathbb{R}^3)$ be such that $-\Delta v_j + q_j v_j \in L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \gamma_j^{1/2}(-\Delta v_j + q_j v_j)\varphi \, dx - 2\int_{\{x^3=0\}} v_j \partial_{x^3}\sigma_j^{1/2}\varphi \, dx' = 0$$

for all $\varphi \in H^1(\mathbb{R}^3)$ with $q_j(x) = (\sigma_j^{-1/2}\Delta\sigma_j^{1/2})(x)$ for $x \in \mathbb{R}^3_-$ and $q_j(x) = (\sigma_j^{-1/2}\Delta\sigma_j^{1/2})(\mathcal{R}(x))$ for $x \in \mathbb{R}^3_+$. Then $u = \gamma_j^{-1/2} v \in H^1(\mathbb{R}^3)$ is a weak solution of $\nabla \cdot (\gamma_j \nabla u) = 0$ in $\mathbb{R}^3$.

The key point to remove the assumption $\partial_N \sigma_1 = \partial_N \sigma_2 = 0$ on $\partial U \setminus \overline{\Gamma}$ is to realize that the exponential growing solutions constructed by Isakov are under the conditions of v_j in Lemma 1.1. The proof of this lemma is straightforward but we include it in Appendix C.

It is also important to solve the same kind of questions in more general contexts, for example in the framework of electromagnetism.

- *Uniqueness*: Consider $\mu_1, \gamma_1, \mu_2, \gamma_2 \in L^\infty(\Omega)$ and let C^1_Γ, C^2_Γ denote their corresponding restricted Cauchy data sets. If we suppose $C^1_\Gamma = C^2_\Gamma$, can we ensure that $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$?

- *Stability*: Does there exist a modulus of continuity b such that for any two pairs $\mu_1, \gamma_1, \mu_2, \gamma_2 \in L^\infty(\Omega)$ the following estimate holds

$$\|\mu_1 - \mu_2\|_{L^\infty(\Omega)} + \|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \le b(\delta_C(C^1_\Gamma, C^2_\Gamma))?$$

Here δ_C stands for the pseudo-metric distance of restricted Cauchy data sets.

An important part of this dissertation is dedicated to generalize the Isakov result for the time-harmonic Maxwell equations. However, our goal is not only to prove uniqueness but also stability. Let us state our main results using local boundary data.

Let Γ be a non-empty proper open subset of $\partial\Omega$, the boundary of Ω. Let ν be the 1-form defined by $\nu = e(N, \cdot)$ with N the outward unit vector normal to $\partial\Omega$ and e the euclidean metric. Given a frequency $\omega > 0$, the *Cauchy data set restricted to* Γ is defined as follows: $(T, S) \in C(\mu, \gamma; \Gamma)$ if and only if $T \in TH_0(\Gamma)$, $S \in TH(\Gamma)$, and there exists a pair $(E, H) \in (H(\Omega; \mathrm{curl}))^2$ solution of (1.2) satisfying $*(\nu \wedge E) = T$ and $*(\nu \wedge H)|_\Gamma = S$.

Definition 1.4 Let μ_1, γ_1 and μ_2, γ_2 be two pairs of coefficients. Consider ω a positive frequency and let C^j_Γ denote $C(\mu_j, \gamma_j; \Gamma)$. Let us define the pseudo-metric distance between the restricted Cauchy data sets C^1_Γ and C^2_Γ as

$$\delta_C(C^1_\Gamma, C^2_\Gamma) = \max_{j \neq k} \sup_{\substack{(T_k,S_k) \in C^k_\Gamma \\ \|T_k\|_{TH_0(\Gamma)}=1}} \inf_{(T_j,S_j) \in C^j_\Gamma} \|(T_j, S_j) - (T_k, S_k)\|_{TH_0(\Gamma)\times TH(\Gamma)} .$$

In order to state our result, we need stable determination of the problem on the boundary. Since this has not been proven yet, we shall introduce the following definitions.

Definition 1.5 Given two constants M, s such that $0 < M$, $0 < s < 1/2$, we shall say that the pair of coefficients μ, γ is *admissible* if they satisfy the following conditions.

(i) *Uniform ellipticity condition.* The coefficients $\gamma, \mu \in C^{1,1}(\overline{\Omega})$ satisfy

$$M^{-1} \leq \mathrm{Re}\,\gamma(x) \qquad M^{-1} \leq \mu(x);$$

for any $x \in \Omega$.

(ii) *A priori bound on the boundary.* The following a priori bound holds on the boundary

$$\|\gamma\|_{C^{0,1}(\overline{\Gamma})} + \|\mu\|_{C^{0,1}(\overline{\Gamma})} < M.$$

(iii) *A priori bound in the interior.* The following a priori bounds hold in the interior

$$\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \leq M, \qquad \|\gamma\|_{H^{2+s}(\Omega)} + \|\mu\|_{H^{2+s}(\Omega)} \leq M.$$

Definition 1.6 Let M, s be the constants given in Definition 1.5 and let ω be a positive frequency. We shall say that a pair μ, γ is in the *class of B-stable coefficients on* Γ *at frequency* ω if μ, γ is an admissible pair and there exists a modulus of continuity B such that, for any other admissible pair $\tilde{\mu}, \tilde{\gamma}$, one has

$$\|\gamma - \tilde{\gamma}\|_{C^{0,1}(\overline{\Gamma})} + \|\mu - \tilde{\mu}\|_{C^{0,1}(\overline{\Gamma})} \leq B\left(\delta_C(C_\Gamma, \tilde{C}_\Gamma)\right),$$
$$\|\nabla(\gamma - \tilde{\gamma})\|_{L^\infty(\Gamma;\mathbb{C}^3)} + \|\nabla(\mu - \tilde{\mu})\|_{L^\infty(\Gamma;\mathbb{C}^3)} \leq B\left(\delta_C(C_\Gamma, \tilde{C}_\Gamma)\right).$$

Here $C_\Gamma, \tilde{C}_\Gamma$ are the Cauchy data sets associated to the pairs μ, γ and $\tilde{\mu}, \tilde{\gamma}$, respectively.

Now, our main result using local boundary data is.

Theorem 4 Let U be either a *suitable partially flat domain* or a *suitable partially spherical domain* and let ω be a positive frequency. Consider μ_1, γ_1 and μ_2, γ_2 any two pairs in the class of B-stable coefficients on Γ at frequency ω, with B satisfying $|r| \leq B(|r|)$ for all $|r| < 1$. Assume that $\partial_N \mu_j = \partial_N \gamma_j = 0$ on $\partial U \setminus \overline{\Gamma}$ with $j = 1, 2$. Then, there exists a constant $C = C(M)$ such that the following estimate holds

$$\|\gamma_1 - \gamma_2\|_{H^1(U)} + \|\mu_1 - \mu_2\|_{H^1(U)} \leq C\,|\log B(\delta_C(C^1_\Gamma, C^2_\Gamma))|^{-\lambda},$$

for some constant λ such that $0 < \lambda < s^2/3$. Here C^1_Γ, C^2_Γ are the restricted Cauchy data sets associated to the pairs μ_1, γ_1 and μ_2, γ_2, respectively.

The exact meaning of *suitable* is explained in Section 4.1.

Once again, it should be possible to prove that any admissible pair is in the class of Hölder-stable coefficients on Γ for any frequency ω, that is, with $B(|r|) = |r|^\alpha$ for $0 < \alpha < 1$. Note that we have obtained the same kind of stability as in the global data case.

From the point of view of applications it might be useful to suppose the coefficients to be equal on the accessible part of the boundary. In this particular case we get the following corollary.

Corollary 5 Let U be either a *suitable partially flat domain* or a *suitable partially spherical domain* and let ω be a positive frequency. Consider μ_1, γ_1 and μ_2, γ_2 any two pairs of admissible coefficients. Assume that

$$\mu_1|_\Gamma = \mu_2|_\Gamma,\ \partial_{x^j}\mu_1|_\Gamma = \partial_{x^j}\mu_2|_\Gamma, \quad \gamma_1|_\Gamma = \gamma_2|_\Gamma,\ \partial_{x^j}\gamma_1|_\Gamma = \partial_{x^j}\gamma_2|_\Gamma$$

and $\partial_N \mu_k = \partial_N \gamma_k = 0$ on $\partial U \setminus \overline{\Gamma}$ with $j = 1, 2, 3$ and $k = 1, 2$. Then, there exists a constant $C = C(M)$ such that the following estimate holds

$$\|\gamma_1 - \gamma_2\|_{H^1(U)} + \|\mu_1 - \mu_2\|_{H^1(U)} \leq C|\log \delta_C(C^1_\Gamma, C^2_\Gamma)|^{-\lambda},$$

for some constant λ such that $0 < \lambda < s^2/3$.

Furthermore, if we follow the proof of Theorem 4 one can state the following uniqueness result.

Theorem 6 Let U be either a *suitable partially flat domain* or a *suitable partially spherical domain* and let ω be a positive frequency. Consider μ_1, γ_1 and μ_2, γ_2 in $C^{1,1}(\overline{U})$ such that

$$\mu_1|_\Gamma = \mu_2|_\Gamma,\ \partial_{x^j}\mu_1|_\Gamma = \partial_{x^j}\mu_2|_\Gamma, \quad \gamma_1|_\Gamma = \gamma_2|_\Gamma,\ \partial_{x^j}\gamma_1|_\Gamma = \partial_{x^j}\gamma_2|_\Gamma,$$

with $j = 1, 2, 3$. If additionally $C^1_\Gamma = C^2_\Gamma$ and $\partial_N \mu_k = \partial_N \gamma_k = 0$ on $\partial U \setminus \overline{\Gamma}$ with $k = 1, 2$, then

$$\mu_1 = \mu_2, \qquad \gamma_1 = \gamma_2$$

in U.

As in the inverse conductivity problem, it should be possible to prove that the coefficients are equal on the accessible part of the boundary Γ whenever $C^1_\Gamma = C^2_\Gamma$.

Note that in our results we need to assume $\partial_N \mu_k = \partial_N \gamma_k = 0$ on $\partial U \setminus \overline{\Gamma}$ with $k = 1, 2$. When one tries to produce an argument similar to that of Lemma 1.1, one encounters boundary terms which are difficult to treat. Possibly, the only way of handling those terms is to assume this extra hypothesis. Actually, those boundary terms are not the natural ones for Maxwell's equations, they appear when transforming these equations into a Schrödinger-type equation.

The proof of Theorem 4 in the partially flat case bases on an integral formula relating the accessible boundary data with the coefficients in the interior by means of solutions for the time-harmonic Maxwell equations vanishing on the unaccessible part of the boundary. Once more, we need to construct exponential growing solutions to extract the information coded in this formula. The construction of this kind of solutions requires the Isakov's argument adapted to our context and to transform the time-harmonic Maxwell equations into a Schrödinger-type equation. The integral formula with these solutions yields information of a semilinear elliptic system satisfied by the coefficients. In order to end up with proof in the partially flat case, we use a Carleman-type estimate. Theorem 4 in the partially spherical case can be reduced to the partially flat case by using the Kelvin transform.

1.3 Bibliographical notes

In this section we discuss some of the works related to the problem treated along this dissertation.

A boundary value problem

Theorem 1 might not be new but we have not found in the literature a proof of the precise statement given here. Thus, we include one for completeness. Nevertheless, a proof in the case where $\partial\Omega$ is of class C^2 can be found in [40]. On the other hand, a proof of this result in the context of non-smooth domains is given in [26] but there σ is assumed to be zero. The main ingredients in our proof are a standard compactness result stated in [45] and the well known analytic Fredholm theory. The possible novelty is the use of a lemma due to Peetre [36]. This abstract result allows us to conclude the compactness of the operators

$$\begin{aligned}(*d)^{-1} &: \mathrm{Rang}\,(*d) \longrightarrow \mathrm{Rang}\,(*d)^* \cap H(\Omega;\mathrm{curl}) \cap H_0(\Omega;\mathrm{div})\\ ((*d)^*)^{-1} &: \mathrm{Rang}\,(*d)^* \longrightarrow \mathrm{Rang}\,(*d) \cap H_0(\Omega;\mathrm{curl}) \cap H(\Omega;\mathrm{div}).\end{aligned}$$

The lack of the smoothness on $\partial\Omega$ force us to use the spaces $H(\Omega;\mathrm{curl})$, $H(\Omega;\mathrm{div})$. So one of the technical complications in this dissertation is the need of using a weak definitions for the tangential and normal traces. However, when the boundary is smooth, at least $C^{1,1}$, it is possible to prove the existence of H^1-solutions for $H^{1/2}$-data. This allows us to describe traces in a strong sense and to use the standard Sobolev spaces. It is well known (see [4], [5] and [14]) that Maxwell's equations may not admit H^1-solutions even with $H^{1/2}$-boundary data, whenever the domain is neither convex nor has $C^{1,1}$-boundary.

Stable determination from boundary data

Before recalling the earlier works in the context of the time-harmonic Maxwell equations, let us emphasize the main results on the isotropic Calderón problem for dimension $n = 3$. In 1980, Calderón used harmonic complex functions to prove the uniqueness of the linearized problem [9]. Later, in 1987 Sylvester and Uhlmann pushed forward Calderón's solutions obtaining exponential growing solutions which behave asymptotically as Calderón's. This solutions allowed them to prove uniqueness for regular conductivities [43]. One year later, those solutions were used in [32] by Nachman to recover the conductivity and in [1] by Alessandrini to prove stable determination.

More than ten years later, experts started to apply those techniques in the study of inverse problems in electromagnetism. Three papers appear in 1992, one by Somersalo, Isaacson and Chenay; other by Colton and Päivärinta and finally one by Sun and Uhlmann. In [40] Somersalo *et al* proved the uniqueness for the linearized inverse boundary value problem. In [13], Colton and Päivärinta proved the uniqueness of the inverse scattering problem in the case where the permeability is constant. Finally, in [42] Sun and Uhlmann proved uniqueness of the inverse boundary value problem assuming the permeability, the permittivity and the conductivity to be sufficiently close to constant.

The main idea in [13] was pushed forward later by Ola, Päivärinta and Somersalo. In [34] they proved the unique recovery of C^3-coefficients γ and μ from boundary data, in the context of $C^{1,1}$-domains. The proof was later simplified in [33]. The spaces used to model the boundary measurements in these previous work are *better* than the ones used in this dissertation. This is a consequence of the lack of smoothness on the boundary of our domain.

The results in [34] and [33] assume the coefficients to be constant on the boundary. In order to complete them, Joshi and McDowall proved in [30] and [21] boundary determination in the case where the boundary is smooth. The more general chiral media was studied in [29]. For a slightly more general approach and more background information, see the review article [35].

In [39] Sarkola studied the relation between the inverse scattering problem and the inverse boundary value problem. Further, Sarkola proved that the former problem can be reduced to the latter one. Sarkola's proof is inspired in [32], where Nachman proved that the inverse scattering problem for the Schrödinger equation can be reduced to an inverse boundary value problem for the same equation. On the other hand, in [15] Hähner gave a proof of the stability for the inverse scattering problem when the magnetic permeability μ is constant and as a consequence he obtained stability for the inverse boundary value problem for constant magnetic permeability when the domain is a ball. Our corollary generalizes the stability result stated in [15], at least, for the inverse boundary value problem.

Inverse boundary value problems for Maxwell's equations in anisotropic media have been studied in time domain and time-harmonic setting (see [24] and [23], respectively).

Stable determination from local data

Two different approaches have been used to attack the inverse conductivity problem from partial boundary data. The first one was proposed by Bukhgeim and Uhlmann in [8] and generalized in [22]. In [16], this method was used to give a log-log-stable determination in the framework of [8]. In this approach there are not any strong geometrical restriction about the domain but the partial measurements have to be taken in the whole boundary. Getting an optimal stability (i. e. a stability with a log-type modulus of continuity) in the context of [8] may be difficult. The stability in the framework of [22] is an open question. The second approach for partial data is the one proposed by Isakov. The optimal stable determination of this problem was stated in [17]. As we have already mentioned, this argument requires a strong restriction on the domain. However, the measurements are localized on the accessible part of the boundary and it is possible to get the optimal stable determination. These two facts are very important from the point of view of applications. For instance, Alessandrini and Vessella proved in [2] that a logarithmic estimate yields Lipschitz stability for some finite dimensional spaces of conductivities.

These two approaches have been extended to systems. In [38] Salo and Tzou followed the spirit of [22] to prove uniqueness in the context of Dirac's equation. Isakov's argument was extended in [10] to the time-harmonic Maxwell equations. The proof given in this dissertation takes some ideas from [23] and it turns out to be more convenient and useful for us than the proof given in [10]. In fact, it avoids the long computations made there to prove the thesis of Theorem 6 and it allows to relax the hypothesis about the domain and the smoothness of the coefficients. In [10] the domain was assumed to be of class $C^{1,1}$ and the coefficients were assumed to be C^4. Besides, a technical hypothesis about the extension of the coefficients had to be supposed.

Chapter 2

Time-harmonic Maxwell equations

"Es mejor no saber lo que se sabe."
Lao Tse

In this chapter we briefly present the basic tools of multivariable calculus used along the dissertation. We recall some notions about the Sobolev and Besov spaces and we introduce some non-standard Sobolev spaces adapted to Maxwell's equations. Finally, we study the well-posedness of a boundary value problem for the time-harmonic Maxwell equations.

2.1 Tools of multivariable calculus

Let $\mathbb{E}$ be the three-dimensional euclidean point space and let its tangent bundle be denoted by $T\mathbb{E}$. Let $\mathcal{T}\mathbb{E}$ be the module of smooth vector fields over the real smooth functions $C^\infty(\mathbb{E};\mathbb{R})$ and define

$$\mathcal{X}\mathbb{E} = \{u + iv : u, v \in \mathcal{T}\mathbb{E}\}.$$

The elements of $\mathcal{X}\mathbb{E}$ will be called complex vector fields. Let the bundle of alternating tensors be denoted by $\Lambda^k T\mathbb{E}$ with $k = 0, 1, 2, 3$. Let $\mathcal{A}^k\mathbb{E}$ be the vector space of differential k-forms and define

$$\Lambda^k\mathbb{E} = \{\omega + i\eta : \omega, \eta \in \mathcal{A}^k\mathbb{E}\}.$$

The elements of $\Lambda^k\mathbb{E}$ will be called complex k-forms. Recall that 0-forms are smooth functions by definition. As it is usual, d and $\wedge$ denote the exterior derivative operator and the exterior product of forms, respectively.

The euclidean metric e induces a volume element denoted by dV, a distance denoted by d_e and a point-wise inner product denoted by $\langle \omega, \eta \rangle$ for any $\omega, \eta \in \mathcal{A}^k\mathbb{E}$

with $k = 0, 1, 2, 3$. Recall that the Hodge star operator is the unique bundle map $* : \Lambda^k T\mathbb{E} \longrightarrow \Lambda^{3-k} T\mathbb{E}$ satisfying

$$\omega \wedge *\eta = \langle \omega, \eta \rangle \, dV.$$

Moreover, $* * \omega = \omega$. Let us define $|\eta|^2 = \langle \eta, \eta \rangle$.

The formal adjoint of d will be denoted by δ and it can be expressed by

$$\delta\eta = (-1)^k * d * \eta$$

for $\eta \in \mathcal{A}^k$. Let us define the laplacian on k-forms as $-\Delta := \delta d + d\delta$.

We also recall that we can identify vectors and 1-forms by means of the metric, that is,

$$u \in \mathcal{T}\mathbb{E} \longmapsto \eta = e(u, \centerdot) \in \mathcal{A}^1\mathbb{E}. \tag{2.1}$$

If $u \in \mathcal{T}\mathbb{E}$, its corresponding 1-form will be denoted by $u^\flat$. However if the difference is clear by the context it will be denoted by u. On the other hand, if $v \in \mathcal{A}^1\mathbb{E}$, its corresponding vector field will be denoted by $v^\sharp$. As before, this notation will be used whenever the context is not clear.

Finally, for any $f \in C^\infty(\mathbb{E}; \mathbb{R})$ and any $u, v \in \mathcal{T}\mathbb{E}$, $u \cdot v = e(u, v)$ denotes the point-wise inner product, $u \times v = *(u^\flat \wedge v^\flat)^\sharp$ denotes the point-wise cross product and $\nabla f = (df)^\sharp$, $\nabla \cdot u = -\delta u^\flat$ and $\nabla \times u = (*du^\flat)^\sharp$ stand for the gradient, divergence and curl, respectively.

Lipschitz domains

Definition 2.1 Let Ω be a nonempty proper open subset of $\mathbb{E}$ and consider a point P_0 on its boundary $\partial\Omega$. We say that Ω is a *Lipschitz domain near* P_0 if there exist

(i) a plane $q \subset \mathbb{E}$ passing through P_0 and a choice of a unit vector N_q normal to q;

(ii) some euclidean coordinates $\mathcal{E}_0 : \mathbb{E} \to \mathbb{E}$ such that $\mathcal{E}_0(P_0)^j = 0$ for $j = 1, 2, 3$ and $\mathcal{E}_0(Q) \in \mathbb{R}^2 \times \{0\}$, for any $Q \in q$ (for short, we shall denote $\mathcal{E}_0(P)^j$ by y^j);

(iii) and an open cylinder $C^{P_0}_{c_1,c_2} = \{P \in \mathbb{E} : |y'| < c_1, |y^3| < c_2\}$ –called coordinate cylinder near P_0– such that

$$\begin{aligned}
C^{P_0}_{c_1,c_2} \cap \Omega &= C^{P_0}_{c_1,c_2} \cap \{P \in \mathbb{E} : y^3 > \phi(y^1, y^2)\},\\
C^{P_0}_{c_1,c_2} \cap \partial\Omega &= C^{P_0}_{c_1,c_2} \cap \{P \in \mathbb{E} : y^3 = \phi(y^1, y^2)\},\\
C^{P_0}_{c_1,c_2} \cap \overline{\Omega}^c &= C^{P_0}_{c_1,c_2} \cap \{P \in \mathbb{E} : y^3 < \phi(y^1, y^2)\};
\end{aligned}$$

for some Lipschitz function $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\phi(0) = 0, \quad \text{and} \quad |\phi(y^1, y^2)| < c_2 \quad \text{if} \quad |y'| \leq c_1.$$

Finally, we say that Ω is a *Lipschitz domain* if it is a Lipschitz domain near every point $P \in \partial\Omega$.

In this definition the superscript c denotes the complement of a set, relative to $\mathbb{E}$; and $|y'|^2 = |y^1|^2 + |y^2|^2$.

Recall that along the boundary of any Lipschitz domain there exists a measurable unit normal vector field N pointing outward. Let ν denote $N^\flat$. In this context, we can set the following integration by parts formulas

$$-\int_\Omega (\nabla \cdot u) f \, dV = \int_\Omega u \cdot \nabla f \, dV - \int_{\partial\Omega} (N \cdot u) f|_{\partial\Omega} \, dA, \tag{2.2}$$

$$\int_\Omega (\delta u) f \, dV = \int_\Omega \langle u, df \rangle \, dV - \int_{\partial\Omega} \langle \nu, u \rangle f|_{\partial\Omega} \, dA, \tag{2.3}$$

$$\int_\Omega (\nabla \times u) \cdot v \, dV = \int_\Omega u \cdot (\nabla \times v) \, dV + \int_{\partial\Omega} (N \times u) \cdot v|_{\partial\Omega} \, dA \tag{2.4}$$

and

$$\int_\Omega \langle *du, v \rangle \, dV = \int_\Omega \langle u, *dv \rangle \, dV + \int_{\partial\Omega} \langle *(\nu \wedge u), v|_{\partial\Omega} \rangle \, dA. \tag{2.5}$$

Here $f \in C^\infty(\overline{\Omega}) = \{f_1|_{\overline{\Omega}} + i f_2|_{\overline{\Omega}} : f_1, f_2 \in C^\infty(\mathbb{E}; \mathbb{R})\}$, $u, v \in \mathcal{X}\mathbb{E}|_\Omega = \{u|_{\overline{\Omega}} : u \in \mathcal{X}\mathbb{E}\}$ in (2.2) and (2.4), and $u, v \in \Lambda^1\mathbb{E}|_\Omega = \{u|_{\overline{\Omega}} : u \in \Lambda^1\mathbb{E}\}$ in (2.3) and (2.5). Moreover, dA stands for the area element induced by the volume element $dA := dV(N, \cdot, \cdot)$ and $\cdot|_{\partial\Omega}, \cdot|_{\overline{\Omega}}$ denote the restrictions to $\partial\Omega$ and $\overline{\Omega}$, respectively.

2.2 Functional spaces

Sobolev and Besov spaces

Let $C_0^\infty(\mathbb{E})$ denote the set of complex smooth functions on $\mathbb{E}$ with compact support. As it is usual, let $L^p(\mathbb{E})$ denote the Lebesgue spaces on $\mathbb{E}$ with $1 \leq p \leq \infty$.

Definition 2.2 For any $s \in \mathbb{R}$, let us define the *potential Sobolev space on* $\mathbb{E}$ as

$$H^s(\mathbb{E}) = \overline{C_0^\infty(\mathbb{E})}^{\|\cdot\|_{H^s(\mathbb{E})}},$$

with the norm

$$\|f\|^2_{H^s(\mathbb{E})} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi.$$

Here $\widehat{f}$ stands for the Fourier transform of f. When $s = 0$ this space is $L^2(\mathbb{E})$.

Along this dissertation, Ω will denote a bounded Lipschitz domain and $L^p(\Omega)$ denote the Lebesgue spaces on Ω. Let $\cdot|_\Omega$ denote the restriction to Ω.

Definition 2.3 For any $s \geq 0$, let us define the *potential Sobolev space on* Ω as

$$H^s(\Omega) = \{f|_\Omega : f \in H^s(\mathbb{E})\},$$

with the norm

$$\|g\|_{H^s(\Omega)} = \inf\{\|f\|_{H^s(\mathbb{E})} : f|_\Omega = g\}.$$

On the other hand, for $s \in \mathbb{R}$, let us define the *potential Sobolev space of functions supported in* Ω as

$$H_0^s(\Omega) = \{f \in H^s(\mathbb{E}) : \operatorname{supp} f \subseteq \overline{\Omega}\},$$

with norm

$$\|f\|_{H_0^s(\Omega)} = \|f\|_{H^s(\mathbb{E})} .$$

Finally, for $s > 0$, define the space $H^{-s}(\Omega)$ as the dual of $H_0^s(\Omega)$, that is,

$$H^{-s}(\Omega) = (H_0^s(\Omega))^*.$$

When $s = 0$ this is $L^2(\Omega)$.

Whenever s is a non-negative integer the spaces $H^s(\mathbb{E}), H^s(\Omega)$ can be identified with the spaces of distributions with k derivatives in $L^2(\mathbb{E}), L^2(\Omega)$ respectively, for $0 \leq k \leq s$. Additionally, it is well known that $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$, for $s \in \mathbb{R}$.

Proposition 2.1 For all $s > 0$, $H_0^{-s}(\Omega)$ is the dual space of $H^s(\Omega)$, that is,

$$H_0^{-s}(\Omega) = (H^s(\Omega))^*.$$

Proof: The proof is an easy exercise and it can be found in [20]. □

Proposition 2.2 The extension by zero outside of Ω allows to identify the spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ for all $-1/2 < s < 1/2$.

Proof: The proof can be found in [44]. □

It is well-known that $H^s(\mathbb{E})$ is a complex interpolation scale for $s \in \mathbb{R}$; that is, for any $s_1, s_2 \in \mathbb{R}$, one has

$$[H^{s_1}(\mathbb{E}), H^{s_2}(\mathbb{E})]_\theta = H^s(\mathbb{E}),$$

with $s = \theta s_1 + (1-\theta)s_2$ with $\theta \in (0, 1)$ (see details in [3]). Then the following estimate is an easy consequence of Proposition 2.2:

$$\|f\|_{H_0^s(\Omega)} \leq C \|f\|^{\theta}_{H_0^{s_1}(\Omega)} \|f\|^{1-\theta}_{H^{s_2}(\Omega)} , \tag{2.6}$$

Definition 2.4 Let F be any closed subset of $\mathbb{E}$. Let us define the Lipschitz space of index α as

$$C^{0,\alpha}(F) = \{f : F \to \mathbb{C}; |f(P)| \leq M, |f(P) - f(Q)| \leq M d_e(P,Q)^\alpha, P, Q \in \mathbb{E}\}.$$

The norm on this space is defined as the smallest constant M.

Recall that, given any $f \in C^{0,\alpha}(F)$, there exists an extension $\tilde{f}$ of f such that $\tilde{f} \in C^{0,\alpha}(\mathbb{E})$ and

$$\left\|\tilde{f}\right\|_{C^{0,\alpha}(\mathbb{E})} \leq C \|f\|_{C^{0,\alpha}(F)},$$

where the constant is independent of F (see details in [41]).

Definition 2.5 Let us define the space $C^{1,1}(\overline{\Omega})$ as

$$C^{1,1}(\overline{\Omega}) = \{f|_{\overline{\Omega}} : \partial^\alpha f \in C^{0,1}(\mathbb{E}), 0 < |\alpha| \leq 1\}.$$

Definition 2.6 Let us define the Sobolev space $W^{1,\infty}(\mathbb{E})$ as

$$W^{1,\infty}(\mathbb{E}) = \{f \in L^\infty(\mathbb{E}) : \partial^\alpha f \in L^\infty(\mathbb{E}),\ 0 < |\alpha| \leq 1\},$$

with the norm

$$\|f\|_{W^{1,\infty}(\mathbb{E})} = \sum_{0 \leq |\alpha| \leq 1} \|\partial^\alpha f\|_{L^\infty(\mathbb{E})}.$$

Recall that the space $C^{0,1}(\mathbb{E})$ is isomorphic to $W^{1,\infty}(\mathbb{E})$.

Definition 2.7 Let us define the Sobolev space $W^{2,\infty}(\Omega)$ as

$$W^{2,\infty}(\Omega) = \{f \in L^\infty(\Omega) : \partial^\alpha f \in L^\infty(\Omega),\ 0 < |\alpha| \leq 2\},$$

with the norm

$$\|f\|_{W^{2,\infty}(\Omega)} = \sum_{0 \leq |\alpha| \leq 2} \|\partial^\alpha f\|_{L^\infty(\Omega)}.$$

We next define the Besov spaces in an intrinsic way. To do it we introduce the functional

$$I_s(f) = \left(\int_{\mathbb{R}^2} \frac{\|f(\cdot + y) - f(\cdot)\|^2_{L^2(\mathbb{R}^2)}}{|y|^{2+2s}}\, dy \right)^{1/2},$$

defined for $f \in \mathcal{S}(\mathbb{R}^2)$ –the space of rapidly decreasing functions.

Definition 2.8 For $0 < s < 1$, let us define the *Besov space* as

$$B^s(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : I^s(f) < +\infty\},$$

with the norm

$$\|f\|_{B^s(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} + I^s(f).$$

Now we shall extend these Besov spaces on $\mathbb{R}^2$ to Besov spaces on $\partial\Omega$. Let $P_1, \ldots, P_n$ belong to $\partial\Omega$ and $\Gamma_1, \ldots, \Gamma_n$ be $\Gamma_j = C^{P_j}_{c1,c_2} \cap \partial\Omega$ for $j = 1, \ldots, n$, such that $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_n$; and consider a partition of unity $\chi_1, \ldots, \chi_n$ subordinate to $\Gamma_1, \ldots, \Gamma_n$. We shall say that $f \in B^s(\partial\Omega)$ for $0 < s < 1$ if

$$(\chi_j f) \circ \mathcal{E}_j^{-1}(\cdot, \phi_j(\cdot)) \in B^s(\mathbb{R}^2),$$

for any possible choice of points and any partition of unity related to them as above. Here $\mathcal{E}_j$ and ϕ_j are, respectively, the euclidean coordinates and the function defining the boundary locally, corresponding to the point P_j. The norm defined on these spaces will be given by

$$\|f\|_{B^s(\partial\Omega)} = \inf\{\sum_{j=1}^{n} \left\|(\chi_j f) \circ \mathcal{E}_j^{-1}(\cdot, \phi_j(\cdot))\right\|_{B^s(\mathbb{R}^2)} : n \in \mathbb{N},$$
$$P_j \in \partial\Omega,\ \mathrm{supp}(\chi_j) \subset \Gamma_j,\ j = 1, \ldots, n\}.$$

One of the reasons to introduce these spaces is to describe the properties of the trace operator.

Proposition 2.3 The trace operator $\cdot|_{\partial\Omega} : H^s(\Omega) \longrightarrow B^{s-1/2}(\partial\Omega)$ is bounded and onto whenever $1/2 < s < 3/2$. Furthermore, it has a bounded right inverse whose norm is controlled by s and the Lipschitz character of Ω.

Proof: A proof of this statement can be found in [20]. □

Definition 2.9 For $0 < s < 1$, let us define the space $B^{-s}(\partial\Omega)$ as the dual of $B^s(\partial\Omega)$, that is,

$$B^{-s}(\partial\Omega) = (B^s(\partial\Omega))^*.$$

Non-standard Sobolev and Besov spaces

We shall introduce some spaces adapted to Maxwell's equations from two different approaches: as vectors fields and as 1-forms.

Vector fields approach. We shall choose euclidean coordinates $\mathcal{E}$ when working with vector fields. We shall identify points $P \in \mathbb{E}$ with their coordinates $x = (x^1, x^2, x^3) \in \mathbb{R}^3$. Moreover, vectors fields will be expressed at each point as

$$u(x) = \begin{pmatrix} u^{(1)} & u^{(2)} & u^{(3)} \end{pmatrix}^t (x) \in \mathbb{C}^3.$$

Definition 2.10 For all $s \in \mathbb{R}$, let us define

$$H^s(\Omega; \mathbb{C}^3) = \overline{\mathcal{X}\mathbb{E}|_\Omega}^{\|\cdot\|_{H^s(\Omega;\mathbb{C}^3)}}$$

where

$$\|u\|^2_{H^s(\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3} \left\| u^{(j)} \right\|^2_{H^s(\Omega)}.$$

When $s = 0$ this space will be denoted by $L^2(\Omega; \mathbb{C}^3)$.

Definition 2.11 For all $0 < |s| < 1$, let us define

$$B^s(\partial\Omega; \mathbb{C}^3) = \overline{\mathcal{X}\mathbb{E}|_{\partial\Omega}}^{\|\cdot\|_{B^s(\partial\Omega;\mathbb{C}^3)}}$$

where

$$\|w\|^2_{B^s(\partial\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3} \left\| w^{(j)} \right\|^2_{B^s(\partial\Omega)}$$

and $\mathcal{X}\mathbb{E}|_{\partial\Omega} = \{u|_{\partial\Omega} : u \in \mathcal{X}\mathbb{E}\}$. In the same way, let us define

$$L^2(\partial\Omega; \mathbb{C}^3) = \overline{\mathcal{X}\mathbb{E}|_{\partial\Omega}}^{\|\cdot\|_{L^2(\partial\Omega;\mathbb{C}^3)}}$$

where

$$\|w\|^2_{L^2(\partial\Omega;\mathbb{C}^3)} = \sum_{j=1}^{3} \left\| w^{(j)} \right\|^2_{L^2(\partial\Omega)}.$$

By using the trace operator component by component, one has that

$$\cdot|_{\partial\Omega} : H^s(\Omega; \mathbb{C}^3) \longrightarrow B^{s-1/2}(\partial\Omega; \mathbb{C}^3)$$

is bounded and onto, whenever $1/2 < s < 3/2$.

Definition 2.12 Define the spaces

$$H(\Omega; \mathrm{div}) = \{u \in L^2(\Omega; \mathbb{C}^3) : \nabla \cdot u \in L^2(\Omega)\},$$
$$H(\Omega; \mathrm{curl}) = \{v \in L^2(\Omega; \mathbb{C}^3) : \nabla \times v \in L^2(\Omega; \mathbb{C}^3)\}$$

equipped with the graph norms

$$\|u\|_{H(\Omega;\mathrm{div})} = \|u\|_{L^2(\Omega;\mathbb{C}^3)} + \|\nabla \cdot u\|_{L^2(\Omega)},$$
$$\|v\|_{H(\Omega;\mathrm{curl})} = \|v\|_{L^2(\Omega;\mathbb{C}^3)} + \|\nabla \times v\|_{L^2(\Omega;\mathbb{C}^3)}.$$

Now we define the traces of elements belonging to these spaces.

Definition 2.13 For any $u \in H(\Omega; \mathrm{div})$ the *normal trace* of u can be defined as an element of $B^{-1/2}(\partial\Omega)$, as follows: for any $g \in B^{1/2}(\partial\Omega)$,

$$\langle N \cdot u | g \rangle = \int_\Omega (\nabla \cdot u) \overline{f} \, dV + \int_\Omega u \cdot \overline{\nabla f} \, dV, \tag{2.7}$$

where $f \in H^1(\Omega)$ and $f|_{\partial\Omega} = g$.

Proposition 2.4 The operator

$$N \cdot \mathbf{.} : H(\Omega; \mathrm{div}) \longrightarrow B^{-1/2}(\partial\Omega)$$

is bounded and onto.

Proof: An easy proof of this proposition can be found in [31]. □

Let the kernel of $N \cdot \mathbf{.}$ be denoted by

$$H_0(\Omega; \mathrm{div}) = \{u \in H(\Omega; \mathrm{div}) : N \cdot u = 0\}.$$

Definition 2.14 For any $u \in H(\Omega; \mathrm{curl})$ the *tangential trace* of u can be defined as an element of $B^{-1/2}(\partial\Omega; \mathbb{C}^3)$. That is, for any $w \in B^{1/2}(\partial\Omega; \mathbb{C}^3)$,

$$\langle N \times u | w \rangle = \int_\Omega (\nabla \times u) \cdot \overline{v} \, dV - \int_\Omega u \cdot (\overline{\nabla \times v}) \, dV,$$

where $v \in H^1(\Omega; \mathbb{C}^3)$ and $v|_{\partial\Omega} = w$.

Unlike the tangential trace operator,

$$N \times \mathbf{.} : H(\Omega; \mathrm{curl}) \longrightarrow B^{-1/2}(\partial\Omega; \mathbb{C}^3)$$

is bounded but not onto. Let its range and its kernel be denoted by

$$TH(\partial\Omega) = \{w \in B^{-1/2}(\partial\Omega; \mathbb{C}^3) : \exists u \in H(\Omega; \mathrm{curl}), \ N \times u = w\}$$

and

$$H_0(\Omega; \mathrm{curl}) = \{u \in H(\Omega; \mathrm{curl}) : N \times u = 0\},$$

respectively.

Proposition 2.5 The following items hold:

(a) The vector space $TH(\partial\Omega)$ equipped with the norm

$$\|w\|_{TH(\partial\Omega)} = \inf\{\|u\|_{H(\Omega;\mathrm{curl})} : u \in H(\Omega; \mathrm{curl}), \ N \times u = w\}$$

is a reflexive Banach space.

(b) $TH(\partial\Omega)$ is continuously embedded into $B^{-1/2}(\partial\Omega;\mathbb{C}^3)$ and

$$N\times\mathcal{X}\mathbb{E}|_\Omega \hookrightarrow TH(\partial\Omega)$$

continuously and densely.

(c) The map

$$N\times\centerdot : TH(\partial\Omega) \longrightarrow (TH(\partial\Omega))^*$$

given by

$$\langle N\times w_1|w_2\rangle = \int_\Omega (\nabla\times u)\cdot\overline{v}\,dV - \int_\Omega u\cdot(\overline{\nabla\times v})\,dV,$$

for $w_1, w_2 \in TH(\partial\Omega)$ with $u, v \in H(\Omega;\text{curl})$ such that $N\times u = w_1$, $N\times v = w_2$, is well-defined, bounded and an isomorphism.

(d) One has that $(N\times\centerdot)^{-1} = -N\times\centerdot$, $(N\times\centerdot)^* = -N\times\centerdot$ and

$$\int_\Omega (\nabla\times u)\cdot\overline{v}\,dV = \int_\Omega u\cdot(\overline{\nabla\times v})\,dV - \langle N\times u|N\times(N\times v)\rangle\,. \tag{2.8}$$

Proof: The proof of this proposition can be found in [31]. □

The surface divergence of elements of $TH(\partial\Omega)$ makes sense as elements of $B^{-1/2}(\partial\Omega)$.

Definition 2.15 Let us define the *surface divergence* operator over the space $TH(\partial\Omega)$

$$\text{Div} : TH(\partial\Omega) \longrightarrow B^{-1/2}(\partial\Omega),$$

as

$$\text{Div}\, w = -N\cdot(\nabla\times u),$$

where $u \in H(\Omega;\text{curl})$ and $N\times u = w$.

Since $\nabla\times u \in H(\Omega;\text{div})$, $-N\cdot(\nabla\times u)$ makes sense and it belongs to $B^{-1/2}(\partial\Omega)$; the surface divergence operator is well-defined and bounded.

Theorem 2.1 (Mitrea [31]) There exist constants $C_1, C_2 > 0$ such that, for any $w \in TH(\partial\Omega)$, the following estimates hold

$$\|w\|_{B^{-1/2}(\partial\Omega;\mathbb{C}^3)} + \|\text{Div}\, w\|_{B^{-1/2}(\partial\Omega)} \leq C_1 \|w\|_{TH(\partial\Omega)}\,, \tag{2.9}$$

$$\|w\|_{TH(\partial\Omega)} \leq C_2 \left(\|w\|_{B^{-1/2}(\partial\Omega;\mathbb{C}^3)} + \|\text{Div}\, w\|_{B^{-1/2}(\partial\Omega)}\right). \tag{2.10}$$

In particular, the space $TH(\partial\Omega)$ can be described as the completion of $N\times\mathcal{X}\mathbb{E}|_\Omega \hookrightarrow TH(\partial\Omega)$ in the norm

$$w \longmapsto \|w\|_{B^{-1/2}(\partial\Omega;\mathbb{C}^3)} + \|\text{Div}\, w\|_{B^{-1/2}(\partial\Omega)}\,.$$

Differential forms approach. Now we do not choose coordinates.

Definition 2.16 For all $s \in \mathbb{R}$, let us define

$$H^s(\Omega; \Lambda^1 T\mathbb{E}) = \overline{\Lambda^1\mathbb{E}|_\Omega}^{\|\cdot\|_{H^s(\Omega;\Lambda^1 T\mathbb{E})}}$$

where

$$\|u\|^2_{H^s(\Omega;\Lambda^1 T\mathbb{E})} = \inf_{\mathcal{E}} \left\|u^\sharp\right\|_{H^s(\Omega;\mathbb{C}^3)}.$$

When $s = 0$ this space will be denoted by $L^2(\Omega; \Lambda^1 T\mathbb{E})$.

Definition 2.17 For all $0 < |s| < 1$, let us define

$$B^s(\partial\Omega; \Lambda^1 T\mathbb{E}) = \overline{\Lambda^1\mathbb{E}|_{\partial\Omega}}^{\|\cdot\|_{B^s(\partial\Omega;\Lambda^1 T\mathbb{E})}}$$

where

$$\|w\|^2_{B^s(\partial\Omega;\Lambda^1 T\mathbb{E})} = \inf_{\mathcal{E}} \left\|w^\sharp\right\|_{B^s(\partial\Omega;\mathbb{C}^3)}$$

and $\Lambda^k\mathbb{E}|_{\partial\Omega} = \{u|_{\partial\Omega} : u \in \Lambda^k\mathbb{E}\}$ with $k = 0, 1, 2, 3$. $\Lambda^0\mathbb{E}|_{\partial\Omega}$ can be also denoted by $C^\infty(\partial\Omega)$. In the same way, let us define

$$L^2(\partial\Omega; \Lambda^1 T\mathbb{E}) = \overline{\Lambda^1\mathbb{E}|_{\partial\Omega}}^{\|\cdot\|_{L^2(\partial\Omega;\Lambda^1 T\mathbb{E})}}$$

where

$$\|w\|_{L^2(\partial\Omega;\Lambda^1 T\mathbb{E})} = \inf_{\mathcal{E}} \left\|w^\sharp\right\|_{L^2(\partial\Omega;\mathbb{C}^3)}.$$

The trace operator

$$\cdot|_{\partial\Omega} : H^s(\Omega; \Lambda^1 T\mathbb{E}) \longrightarrow B^{s-1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$$

is bounded and onto, whenever $1/2 < s < 3/2$.

Definition 2.18 Define the spaces

$$H(\Omega; \mathrm{div}) = \{u \in L^2(\Omega; \Lambda^1 T\mathbb{E}) : \delta u \in L^2(\Omega)\},$$
$$H(\Omega; \mathrm{curl}) = \{v \in L^2(\Omega; \Lambda^1 T\mathbb{E}) : *dv \in L^2(\Omega; \Lambda^1 T\mathbb{E})\}$$

equipped with the graph norms

$$\|u\|_{H(\Omega;\mathrm{div})} = \|u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} + \|\delta u\|_{L^2(\Omega)},$$
$$\|v\|_{H(\Omega;\mathrm{curl})} = \|v\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} + \|*dv\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}.$$

Now we define the traces of elements belonging to these spaces.

Definition 2.19 For any $u \in H(\Omega; \mathrm{div})$ the *normal trace* of u can be defined as an element of $B^{-1/2}(\partial\Omega)$. That is, for any $g \in B^{1/2}(\partial\Omega)$,

$$\langle\langle \nu, u\rangle | g\rangle = -\int_\Omega (\delta u)\overline{f}\, dV + \int_\Omega \langle u, \overline{df}\rangle\, dV, \tag{2.11}$$

where $f \in H^1(\Omega)$ and $f|_{\partial\Omega} = g$.

Proposition 2.6 The operator

$$\langle \nu, \mathbf{.}\rangle : H(\Omega; \mathrm{div}) \longrightarrow B^{-1/2}(\partial\Omega)$$

is bounded and onto.

Proof: An easy proof of this proposition can be found in [31]. □
Let the kernel of $\langle \nu, \mathbf{.}\rangle$ be denoted by

$$H_0(\Omega; \mathrm{div}) = \{u \in H(\Omega; \mathrm{div}) : \langle \nu, u\rangle = 0\}.$$

Definition 2.20 For any $u \in H(\Omega; \mathrm{curl})$ the *tangential trace* of u can be defined as an element of $B^{-1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$. That is, for any $w \in B^{1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$,

$$\langle *(\nu \wedge u) | w\rangle = \int_\Omega \langle (*du), \overline{v}\rangle\, dV - \int_\Omega \langle u, (\overline{*dv})\rangle\, dV,$$

where $v \in H^1(\Omega; \Lambda^1 T\mathbb{E})$ and $v|_{\partial\Omega} = w$.

Unlike the tangential trace operator,

$$*(\nu \wedge \mathbf{.}) : H(\Omega; \mathrm{curl}) \longrightarrow B^{-1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$$

is bounded but not onto. Let its range and its kernel be denoted by

$$TH(\partial\Omega) = \{w \in B^{-1/2}(\partial\Omega; \Lambda^1 T\mathbb{E}) : \exists u \in H(\Omega; \mathrm{curl}),\ *(\nu \wedge u) = w\}$$

and

$$H_0(\Omega; \mathrm{curl}) = \{u \in H(\Omega; \mathrm{curl}) : *(\nu \wedge u) = 0\},$$

respectively.

Proposition 2.7 The following items hold:

(a) The vector space $TH(\partial\Omega)$ equipped with the norm

$$\|w\|_{TH(\partial\Omega)} = \inf\{\|u\|_{H(\Omega;\mathrm{curl})} : u \in H(\Omega; \mathrm{curl}),\ *(\nu \wedge u) = w\}$$

is a reflexive Banach space.

(b) $TH(\partial\Omega)$ is continuously embedded into $B^{-1/2}(\partial\Omega;\Lambda^1 T\mathbb{E})$ and

$$*(\nu\wedge\Lambda^1\mathbb{E}|_\Omega)\hookrightarrow TH(\partial\Omega)$$

continuously and densely.

(c) The map

$$*(\nu\wedge\,\boldsymbol{.}\,):TH(\partial\Omega)\longrightarrow(TH(\partial\Omega))^*$$

given by

$$\langle *(\nu\wedge w_1)|w_2\rangle=\int_\Omega\langle(*du),\overline{v}\rangle\,dV-\int_\Omega\langle u,(\overline{*dv})\rangle\,dV,$$

for $w_1,w_2\in TH(\partial\Omega)$ with $u,v\in H(\Omega;\mathrm{curl})$ such that $*(\nu\wedge u)=w_1$, $*(\nu\wedge v)=w_2$, is well-defined, bounded and an isomorphism.

(d) One has that $(*(\nu\wedge\,\boldsymbol{.}\,))^{-1}=-*(\nu\wedge\,\boldsymbol{.}\,)$, $(*(\nu\wedge\,\boldsymbol{.}\,))^*=-*(\nu\wedge\,\boldsymbol{.}\,)$ and

$$\int_\Omega\langle *du,\overline{v}\rangle\,dV=\int_\Omega\langle u,\overline{*dv}\rangle\,dV-\langle *(\nu\wedge u)|*(\nu\wedge*(\nu\wedge v))\rangle\,. \tag{2.12}$$

Proof: The proof of this proposition can be found in [31]. □

The surface divergence of elements of $TH(\partial\Omega)$ makes sense as elements of $B^{-1/2}(\partial\Omega)$.

Definition 2.21 Let us define the *surface divergence* operator over the space $TH(\partial\Omega)$

$$\mathrm{Div}:TH(\partial\Omega)\longrightarrow B^{-1/2}(\partial\Omega),$$

as

$$\mathrm{Div}\,w=-\langle\nu,*du\rangle,$$

where $u\in H(\Omega;\mathrm{curl})$ and $*(\nu\wedge u)=w$.

Since $*du\in H(\Omega;\mathrm{div})$, $-\langle\nu,*du\rangle$ makes sense and it belongs to $B^{-1/2}(\partial\Omega)$; the surface divergence operator is well-defined and bounded.

Theorem 2.2 (Mitrea [31]) There exist constants $C_1,C_2>0$ such that, for any $w\in TH(\partial\Omega)$, the following estimates hold

$$\|w\|_{B^{-1/2}(\partial\Omega;\Lambda^1T\mathbb{E})}+\|\mathrm{Div}\,w\|_{B^{-1/2}(\partial\Omega)}\le C_1\|w\|_{TH(\partial\Omega)}\,, \tag{2.13}$$

$$\|w\|_{TH(\partial\Omega)}\le C_2\left(\|w\|_{B^{-1/2}(\partial\Omega;\Lambda^1T\mathbb{E})}+\|\mathrm{Div}\,w\|_{B^{-1/2}(\partial\Omega)}\right), \tag{2.14}$$

In particular, the space $TH(\partial\Omega)$ can be described as the completion of $*(\nu\wedge\Lambda^1\mathbb{E}|_\Omega)\hookrightarrow TH(\partial\Omega)$ in the norm

$$w\longmapsto\|w\|_{B^{-1/2}(\partial\Omega;\Lambda^1T\mathbb{E})}+\|\mathrm{Div}\,w\|_{B^{-1/2}(\partial\Omega)}\,.$$

Some remarks on the boundary

Along this section, Ω denotes any bounded Lipschitz domain, $\partial\Omega$ denotes its boundary and Γ stands for a proper non-empty open subset of $\partial\Omega$. Moreover, $\cdot|_\Gamma$ and $\cdot|_{\overline{\Gamma}}$ denote the restrictions to Γ and $\overline{\Gamma}$, respectively.

Definition 2.22 For $0 < s < 1$, define the space $B^s(\Gamma)$ as the space

$$B^s(\Gamma) = \{f|_\Gamma : f \in B^s(\partial\Omega)\},$$

with the norm

$$\|g\|_{B^s(\Gamma)} = \inf\{\|f\|_{B^s(\partial\Omega)} : f|_\Gamma = g\}.$$

On the other hand, for $0 < |s| < 1$, define

$$B_0^s(\Gamma) = \{f \in B^s(\partial\Omega) : \operatorname{supp} f \subset \overline{\Gamma}\},$$

with norm

$$\|f\|_{B_0^s(\Gamma)} = \|f\|_{B^s(\partial\Omega)} .$$

Finally, for $0 < s < 1$, define the space $B^{-s}(\Gamma)$ as the dual of $B_0^s(\Gamma)$, that is,

$$B^{-s}(\Gamma) = (B_0^s(\Gamma))^*.$$

Proposition 2.8 For $0 < s < 1$, $B_0^{-s}(\Gamma)$ is the dual space of $B^s(\Gamma)$, that is,

$$B_0^{-s}(\Gamma) = (B^s(\Gamma))^*.$$

Proof: Given $f \in B_0^{-s}(\Gamma)$ we can define $l : B^s(\Gamma) \to \mathbb{C}$ as $l(g) = \langle f|\tilde{g}\rangle$, for any $g \in B^s(\Gamma)$ and $\tilde{g} \in B^s(\partial\Omega)$ such that $\tilde{g}|_\Gamma = g$. Since $\operatorname{supp} f \subset \overline{\Gamma}$ the definition does not depend on the choice of the extension of g. Moreover,

$$|l(g)| = |\langle f|\tilde{g}\rangle| \leq \|f\|_{B_0^{-s}(\Gamma)} \|\tilde{g}\|_{B^s(\partial\Omega)} ,$$

which implies

$$|l(g)| \leq \|f\|_{B_0^{-s}(\Gamma)} \|g\|_{B^s(\Gamma)} ,$$

hence

$$\|l\|_{(B^s(\Gamma))^*} \leq \|f\|_{B_0^{-s}(\Gamma)} .$$

Conversely, given a bounded linear functional $l : B^s(\Gamma) \to \mathbb{C}$ we can construct another functional $\tilde{l} : B^s(\partial\Omega) \to \mathbb{C}$ defined by $\tilde{l}(g) = l(g|_\Gamma)$. Since $\tilde{l}$ is linear and bounded, there exists $f \in B^{-s}(\partial\Omega)$ such that $\langle f|g\rangle = \tilde{l}(g) = l(g|_\Gamma)$. Note that $\operatorname{supp} f \subset \overline{\Gamma}$ and

$$|\langle f|g\rangle| = |l(g|_\Gamma)| \leq \|l\|_{(B^s(\Gamma))^*} \|g|_\Gamma\|_{B^s(\Gamma)} ,$$

which implies

$$|\langle f|g\rangle| \leq \|l\|_{(B^s(\Gamma))^*} \|g\|_{B^s(\partial\Omega)},$$

hence,

$$\|f\|_{B_0^{-s}(\Gamma)} \leq \|l\|_{(B^s(\Gamma))^*}.$$

$\square$

Lemma 2.1 Let s, ϵ be such that $0 < s < 1$ and $0 < \epsilon \leq 1 - s$. Then there exists a constant $C(s,\epsilon) > 0$ such that,

(a) for any $g \in C^{0,s+\epsilon}(\partial\Omega)$ and any $f \in B^s(\partial\Omega)$,

$$\|gf\|_{B^s(\partial\Omega)} \leq C \|g\|_{C^{0,s+\epsilon}(\partial\Omega)} \|f\|_{B^s(\partial\Omega)}; \tag{2.15}$$

(b) for any $g : \partial\Omega \longrightarrow \mathbb{C}$ with $g \in C^{0,s+\epsilon}(\overline{\Gamma})$ and any $f \in B_0^s(\Gamma)$,

$$\|gf\|_{B_0^s(\Gamma)} \leq C \|g\|_{C^{0,s+\epsilon}(\overline{\Gamma})} \|f\|_{B_0^s(\Gamma)}; \tag{2.16}$$

(c) for any $g : \partial\Omega \longrightarrow \mathbb{C}$ with $g \in C^{0,s+\epsilon}(\overline{\Gamma})$ and any $f \in B^s(\Gamma)$,

$$\|g|_\Gamma f\|_{B^s(\Gamma)} \leq C \|g\|_{C^{0,s+\epsilon}(\overline{\Gamma})} \|f\|_{B^s(\Gamma)}. \tag{2.17}$$

Remark: The constant $C(s,\epsilon)$ given here blows up when ϵ becomes small.

Proof: We start proving (a). Let $P_1, \ldots, P_n$ belong to $\partial\Omega$ and $\Gamma_1, \ldots, \Gamma_n$ be such that $\Gamma_j = C_{c_1,c_2}^{P_j} \cap \partial\Omega$ for $j = 1, \ldots, n$, such that $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_n$. Let $\lambda > 0$ be the Lebesgue number associated to $\{\Gamma_j\}_{j=1}^n$ and define

$$\tilde{\Gamma}_j = \{P \in \Gamma_j : \inf_{Q \in \partial\Omega \setminus \Gamma_j} d_e(P,Q) > \lambda/4\}.$$

Note that $\partial\Omega = \tilde{\Gamma}_1 \cup \cdots \cup \tilde{\Gamma}_n$. Let us consider a partition of unity $\chi_1, \ldots, \chi_n$ subordinated to $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_n$ and denote

$$f_j(y) = (\chi_j f) \circ \mathcal{E}_j^{-1}(y, \phi_j(y)), \qquad g_j(y) = (\mathbf{1}_{\Gamma_j} g) \circ \mathcal{E}_j^{-1}(y, \phi_j(y)).$$

Here $\mathbf{1}_{\Gamma_j}$ stands for the indicator function of Γ_j. Consider $P_k \in \Gamma_j$ with $k = 1, 2$, we can write $P_k = \mathcal{E}_j^{-1}(y_k, \phi_j(y_k))$ for $y_k \in \mathbb{R}^2$. By the Lipschitz character of ϕ_j, one has

$$d_e(P_1, P_2) \leq C|y_1 - y_2|.$$

Then, noting that

$$(I^s(g_j f_j))^2 \leq 2 \sup_{\substack{x \in \mathcal{E}(\tilde{\Gamma}_j) \\ |y| < \lambda/5}} \frac{|g_j(x+y) - g_j(x)|^2}{|y|^{2(s+\epsilon)}} \|f_j\|_{L^2(\mathbb{R}^2)}^2 \int_{|y|<\lambda/5} \frac{1}{|y|^{2(1-\epsilon)}}\, dy +$$

$$+8\,\|g_j\|^2_{L^\infty(\mathbb{R}^2)}\,\|f_j\|^2_{L^2(\mathbb{R}^2)}\int_{|y|\geq\lambda/5}\frac{1}{|y|^{2(1+s)}}\,dy+2\,\|g_j\|^2_{L^\infty(\mathbb{R}^2)}\left(I^s(f_j)\right)^2,$$

we can achieve the result.

Now (b) and (c) follow easily from (a) taking $\tilde{g}\in C^{0,s+\epsilon}(\partial\Omega)$ an extension of $g|_{\overline{\Gamma}}$ such that

$$\|\tilde{g}\|_{C^{0,s+\epsilon}(\partial\Omega)}\leq C\,\|g\|_{C^{0,s+\epsilon}(\overline{\Gamma})}\,.$$

□

In the same conditions as in Lemma 2.1, one has that

$$\|gf\|_{B^{-s}(\partial\Omega)}\leq C\,\|g\|_{C^{0,s+\epsilon}(\partial\Omega)}\,\|f\|_{B^{-s}(\partial\Omega)}$$

by duality –recall that $\langle gf|h\rangle=\langle f|\overline{g}h\rangle$ for any $h\in B^s(\partial\Omega)$. Moreover, for $0<|s|<1$ and $0<\epsilon\leq 1-|s|$,

$$\|gw\|_{B^s(\partial\Omega;\mathbb{C}^3)}\leq C\,\|g\|_{C^{0,|s|+\epsilon}(\partial\Omega)}\,\|w\|_{B^s(\partial\Omega;\mathbb{C}^3)}\,,$$
$$\|gw\|_{B^s(\partial\Omega;\Lambda^1T\mathbb{E})}\leq C\,\|g\|_{C^{0,|s|+\epsilon}(\partial\Omega)}\,\|w\|_{B^s(\partial\Omega;\Lambda^1T\mathbb{E})}\,.$$

The constants above are the same as the one in Lemma 2.1 and, once again, they blow up when ϵ becomes small.

Motivated by last lemma, let us remark the following facts. If $u\in H(\Omega;\mathrm{div})$, $f\in C^{0,1}(\partial\Omega)$ and $\tilde{f}$ is any extension of f such that $\tilde{f}\in C^{0,1}(\overline{\Omega})$, then

$$\langle fN\cdot u|g\rangle=\langle N\cdot u\big|g\overline{f}\rangle=\left\langle N\cdot(\tilde{f}u)\Big|g\right\rangle, \tag{2.18}$$

for any $g\in B^{1/2}(\partial\Omega)$.

Additionally, if $w\in TH(\partial\Omega)$, $f\in C^{0,1}(\partial\Omega)$ and $\tilde{f}$ is any extension of f such that $\tilde{f}\in C^{0,1}(\overline{\Omega})$, then

$$\langle fw|z\rangle=\langle w\big|\overline{f}z\rangle=\left\langle N\times(\tilde{f}u)\Big|z\right\rangle,$$

for any $z\in B^{1/2}(\partial\Omega;\mathbb{C}^3)$ and any $u\in H(\Omega;\mathrm{curl})$ such that $N\times u=w$. Note that last pairing implies that $fw\in TH(\partial\Omega)$.

Finally, if $w\in TH(\partial\Omega)$, $f\in C^{0,1}(\partial\Omega)$ and $\tilde{f}$ is any extension of f such that $\tilde{f}\in C^{0,1}(\overline{\Omega})$, then

$$\langle fw|N\times z\rangle=\left\langle N\times(\tilde{f}u)\Big|N\times z\right\rangle=\langle w\big|N\times(\overline{f}z)\rangle, \tag{2.19}$$

for any $z\in TH(\partial\Omega)$ and any $u\in H(\Omega;\mathrm{curl})$ such that $N\times u=w$.

Definition 2.23 Define the space $TH(\Gamma)$ as the space

$$TH(\Gamma)=\{w|_\Gamma : w\in TH(\partial\Omega)\},$$

with the norm

$$\|z\|_{TH(\Gamma)} = \inf\{\|w\|_{TH(\partial\Omega)} : w|_\Gamma = z\}.$$

On the other hand, define

$$TH_0(\Gamma) = \{w \in TH(\partial\Omega) : \operatorname{supp} w \subset \overline{\Gamma}\},$$

with norm

$$\|w\|_{TH_0(\Gamma)} = \|w\|_{TH(\partial\Omega)} .$$

Lemma 2.2 Let N be the outward unit vector normal to $\partial\Omega$ and let ν be its associated 1-form. Then

$$N \times TH_0(\Gamma) = (TH(\Gamma))^*, \qquad *(\nu \wedge TH_0(\Gamma)) = (TH(\Gamma))^*.$$

Proof: Here we prove the first identity. The second one can be checked either following the argument below or using the correspondence between vector fields and 1-forms.

Let $l : TH(\Gamma) \to \mathbb{C}$ be a bounded linear functional, we can construct another functional $\tilde{l} : TH(\partial\Omega) \to \mathbb{C}$ defined by $\tilde{l}(w) = l(w|_\Gamma)$, for any $w \in TH(\partial\Omega)$. Since $\tilde{l}$ is linear, bounded and

$$\left\|\tilde{l}\right\|_{(TH(\partial\Omega))^*} \leq \|l\|_{(TH(\Gamma))^*} ,$$

there exists $z \in TH(\partial\Omega)$ such that $\langle N \times z | w\rangle = \tilde{l}(w) = l(w|_\Gamma)$ with

$$\|z\|_{TH(\partial\Omega)} \leq \|l\|_{(TH(\Gamma))^*} .$$

It is clear that $\operatorname{supp} N \times z \subset \overline{\Gamma}$, hence $z \in TH_0(\Gamma)$ and

$$\|z\|_{TH_0(\Gamma)} \leq \|l\|_{(TH(\Gamma))^*} .$$

Conversely, given $z \in TH_0(\Gamma)$ we can define $l : TH(\Gamma) \to \mathbb{C}$ as $l(w) = \langle N \times z | \tilde{w}\rangle$, for any $w \in TH(\Gamma)$ and $\tilde{w} \in TH(\partial\Omega)$ such that $\tilde{w}|_\Gamma = w$. It is well-defined since $\operatorname{supp} N \times z \subset \overline{\Gamma}$. Moreover,

$$|l(w)| \leq \|z\|_{TH_0(\Gamma)} \|\tilde{w}\|_{TH(\partial\Omega)} ,$$

which implies

$$|l(w)| \leq \|z\|_{TH_0(\Gamma)} \|w\|_{TH(\Gamma)} .$$

Therefore, l is a bounded linear operator with norm

$$\|l\|_{(TH(\Gamma))^*} \leq \|z\|_{TH_0(\Gamma)} .$$

□

Lemma 2.3 There exists a positive constant C such that:

(a) For any $w \in TH(\partial\Omega)$ and any $f \in C^{0,1}(\partial\Omega)$, one has that

$$\|fw\|_{TH(\partial\Omega)} \leq C\,\|f\|_{C^{0,1}(\partial\Omega)}\,\|w\|_{TH(\partial\Omega)}\,. \tag{2.20}$$

(b) For any $w \in TH_0(\partial\Omega)$ and any $f : \partial\Omega \longrightarrow \mathbb{C}$ such that $f \in C^{0,1}(\overline{\Gamma})$, one has that

$$\|fw\|_{TH_0(\Gamma)} \leq C\,\|f\|_{C^{0,1}(\overline{\Gamma})}\,\|w\|_{TH_0(\Gamma)}\,. \tag{2.21}$$

Proof: We start proving (a). Consider $\tilde{f}$ an extension of f such that $\tilde{f} \in C^{0,1}(\overline{\Omega})$ satisfying

$$\left\|\tilde{f}\right\|_{C^{0,1}(\overline{\Omega})} \leq C\,\|f\|_{C^{0,1}(\partial\Omega)}$$

and any $u \in H(\Omega;\mathrm{curl})$ such that $N\cdot u = w$. Then

$$\begin{aligned}\|fw\|_{TH(\partial\Omega)} &\leq \left\|\tilde{f}u\right\|_{H(\Omega;\mathrm{curl})} \leq \left\|\tilde{f}\right\|_{C^{0,1}(\overline{\Omega})}\|u\|_{H(\Omega;\mathrm{curl})}\\ &\leq C\,\|f\|_{C^{0,1}(\partial\Omega)}\,\|u\|_{H(\Omega;\mathrm{curl})}\,.\end{aligned}$$

Taking infimum in $u \in H(\Omega;\mathrm{curl})$ one gets the estimate.

Now (b) follows easily from (a) taking and extension $\tilde{f}$ of $f|_{\overline{\Gamma}}$ such that $\tilde{f} \in C^{0,1}(\partial\Omega)$ and satisfying

$$\left\|\tilde{f}\right\|_{C^{0,1}(\partial\Omega)} \leq C\,\|f\|_{C^{0,1}(\overline{\Gamma})}\,.$$

□

Lemma 2.4 The following items hold:

(a) If $w \in TH_0(\Gamma)$, then $\mathrm{Div}\,w \in B_0^{-1/2}(\Gamma)$ and

$$\|\mathrm{Div}\,w\|_{B_0^{-1/2}(\Gamma)} \leq C\,\|w\|_{TH_0(\Gamma)}\,; \tag{2.22}$$

(b) If $z \in TH(\partial\Omega)$, then, for any $f \in B_0^{1/2}(\Gamma)$ and $\tilde{z} \in TH(\partial\Omega)$ such that $\tilde{z}|_\Gamma = z|_\Gamma$, one has $\langle(\mathrm{Div}\,z)|_\Gamma|f\rangle = \langle \mathrm{Div}\,\tilde{z}|f\rangle$ and

$$\|(\mathrm{Div}\,z)|_\Gamma\|_{B^{-1/2}(\Gamma)} \leq C\,\|z|_\Gamma\|_{TH(\Gamma)}\,. \tag{2.23}$$

Proof: It is easy to check both items.

(a) $\mathrm{Div}\,w$ is well-defined and belongs to $B^{-1/2}(\partial\Omega)$. It remains to prove that $\mathrm{supp}\,\mathrm{Div}\,w \subset \overline{\Gamma}$. In order to verify this last point, we just need to have in mind the following facts: if $f \in H^1(\Omega)$, then $N \times \nabla f \in TH(\partial\Omega)$; moreover

$$\operatorname{supp} N \times \nabla f \subset \operatorname{supp} f|_{\partial\Omega}.$$

Indeed, $\operatorname{supp} \operatorname{Div} w \subset \overline{\Gamma}$ since

$$\langle \operatorname{Div} w | f|_{\partial\Omega} \rangle = \langle w | N \times (N \times \nabla f) \rangle \, .$$

The estimate is now immediate using either (2.9) or (2.13).

(b) By an analogous argument to the one given in (a), we have that if $\tilde{z}|_\Gamma = z|_\Gamma$ then

$$(\operatorname{Div} \tilde{z})|_\Gamma = (\operatorname{Div} z)|_\Gamma.$$

Hence the identity follows. The estimate is a consequence of the identity and (2.9) or (2.13).

□

2.3 A boundary value problem

The proof of Theorem 1 presented here is based on two important results. The first one comes from the analytic Fredholm theory and the second one is a pair of compactness embeddings due to Weber.

Theorem 2.3 (Weber [45]) Consider the spaces

$$H_0(\Omega; \mathrm{curl}) \cap H(\Omega; \mathrm{div}), \qquad H(\Omega; \mathrm{curl}) \cap H_0(\Omega; \mathrm{div})$$

equipped with the norm

$$u \longmapsto \|u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} + \|\delta u\|_{L^2(\Omega)} + \|{*du}\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} \, .$$

Then the embeddings

$$H_0(\Omega; \mathrm{curl}) \cap H(\Omega; \mathrm{div}) \overset{c}{\hookrightarrow} L^2(\Omega; \Lambda^1 T\mathbb{E}), \tag{2.24}$$

$$H(\Omega; \mathrm{curl}) \cap H_0(\Omega; \mathrm{div}) \overset{c}{\hookrightarrow} L^2(\Omega; \Lambda^1 T\mathbb{E}) \tag{2.25}$$

are compact.

In order to state the proof of Theorem 1 we shall study the unbounded operator $*d$ in $L^2(\Omega; \Lambda^1 T\mathbb{E})$.

The operator $*d$ in $L^2(\Omega; \Lambda^1 T\mathbb{E})$

Consider the following unbounded operator in $L^2(\Omega; \Lambda^1 T\mathbb{E})$

$$\begin{aligned} *d : &\mathrm{Dom}\,(*d) \subset L^2(\Omega; \Lambda^1 T\mathbb{E}) \longrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \\ &u \in \mathrm{Dom}\,(*d) \longmapsto *du \in L^2(\Omega; \Lambda^1 T\mathbb{E}) \end{aligned} \tag{2.26}$$

with domain $\mathrm{Dom}\,(*d) = H(\Omega; \mathrm{curl})$. Let $\mathrm{Ker}\,(*d)$ and $\mathrm{Rang}\,(*d)$ denote the kernel and the range of $*d$, respectively.

Using (2.12), it can be easily checked that its adjoint is

$$\begin{aligned} (*d)^* : &\mathrm{Dom}\,(*d)^* \subset L^2(\Omega; \Lambda^1 T\mathbb{E}) \longrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \\ &u \in \mathrm{Dom}\,(*d)^* \longmapsto *du \in L^2(\Omega; \Lambda^1 T\mathbb{E}) \end{aligned} \tag{2.27}$$

with domain $\mathrm{Dom}\,(*d)^* = H_0(\Omega; \mathrm{curl})$. Let $\mathrm{Ker}\,(*d)^*$ and $\mathrm{Rang}\,(*d)^*$ denote the kernel and the range of $(*d)^*$, respectively.

Proposition 2.9 $\mathrm{Rang}\,(*d)$ and $\mathrm{Rang}\,(*d)^*$ are closed in $L^2(\Omega; \Lambda^1 T\mathbb{E})$. In fact,

$$\begin{aligned} L^2(\Omega; \Lambda^1 T\mathbb{E}) &= \mathrm{Rang}\,(*d)^* \oplus \mathrm{Ker}\,(*d), \\ L^2(\Omega; \Lambda^1 T\mathbb{E}) &= \mathrm{Rang}\,(*d) \oplus \mathrm{Ker}\,(*d)^*. \end{aligned}$$

Moreover, the $\|\cdot\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}$-bounded operators

$$\begin{aligned} (*d)^{-1} &: \mathrm{Rang}\,(*d) \longrightarrow \mathrm{Rang}\,(*d)^* \cap H(\Omega; \mathrm{curl}) \cap H_0(\Omega; \mathrm{div}) \\ ((*d)^*)^{-1} &: \mathrm{Rang}\,(*d)^* \longrightarrow \mathrm{Rang}\,(*d) \cap H_0(\Omega; \mathrm{curl}) \cap H(\Omega; \mathrm{div}) \end{aligned}$$

are compact.

Proof: Define the $(\|\cdot\|_{H(\Omega;\mathrm{curl})} \to \|\cdot\|_{L^2(\Omega;\mathbb{C}^3)})$-bounded operators

$$\begin{aligned} \mathbf{curl} : &H(\Omega; \mathrm{curl}) \cap \{u \in H_0(\Omega; \mathrm{div}) : \delta u = 0\} \longrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \\ &u \in H(\Omega; \mathrm{curl}) \cap \{u \in H_0(\Omega; \mathrm{div}) : \delta u = 0\} \longmapsto *du \in L^2(\Omega; \Lambda^1 T\mathbb{E}), \\ \mathbf{curl}^* : &H_0(\Omega; \mathrm{curl}) \cap \{u \in H(\Omega; \mathrm{div}) : \delta u = 0\} \longrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \\ &u \in H_0(\Omega; \mathrm{curl}) \cap \{u \in H(\Omega; \mathrm{div}) : \delta u = 0\} \longmapsto *du \in L^2(\Omega; \Lambda^1 T\mathbb{E}). \end{aligned}$$

Here * does not mean that $\mathbf{curl}^*$ is the adjoint of $\mathbf{curl}$. It is just notation.

On the other hand, the $(\|\cdot\|_{H(\Omega;\mathrm{curl})} \to \|\cdot\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})})$-embeddings

$$\begin{aligned} \iota_{\tan} : &H(\Omega; \mathrm{curl}) \cap \{u \in H_0(\Omega; \mathrm{div}) : \delta u = 0\} \hookrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \\ \iota_{\mathrm{nor}} : &H_0(\Omega; \mathrm{curl}) \cap \{u \in H(\Omega; \mathrm{div}) : \delta u = 0\} \hookrightarrow L^2(\Omega; \Lambda^1 T\mathbb{E}) \end{aligned}$$

are compact by (2.25) and (2.24), respectively.

Now, from Peetre's lemma (see Section A.1 for the exact statement) it follows that:

(a) $\operatorname{Ker}(\mathbf{curl})$ and $\operatorname{Ker}(\mathbf{curl}^*)$ are finite dimensional, and $\operatorname{Rang}(\mathbf{curl})$ and $\operatorname{Rang}(\mathbf{curl}^*)$ are closed in $L^2(\Omega;\Lambda^1 T\mathbb{E})$.

(b) There exists a constant $C>0$ such that the following estimates hold

$$\inf_{w\in\operatorname{Ker}(\mathbf{curl})}\|u+w\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}+\|{*du}\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\le C\,\|\mathbf{curl}\,u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\,,$$

for any $u\in H(\Omega;\mathrm{curl})\cap\{u\in H_0(\Omega;\mathrm{div}):\delta u=0\}$; and

$$\inf_{w\in\operatorname{Ker}(\mathbf{curl}^*)}\|u+w\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}+\|{*du}\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\le C\,\|\mathbf{curl}^*\,u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\,,$$

for any $u\in H_0(\Omega;\mathrm{curl})\cap\{u\in H(\Omega;\mathrm{div}):\delta u=0\}$.

In order to show that $\operatorname{Rang}(*d)$ and $\operatorname{Rang}(*d)^*$ are closed in $L^2(\Omega;\Lambda^1 T\mathbb{E})$, it is enough to prove the identities

$$\operatorname{Rang}(*d)=\operatorname{Rang}(\mathbf{curl}),\qquad \operatorname{Rang}(*d)^*=\operatorname{Rang}(\mathbf{curl}^*),$$

since (a) holds. Let $v\in\operatorname{Rang}(*d)$ and $v^*\in\operatorname{Rang}(*d)^*$, then there exists, respectively, $u\in H(\Omega;\mathrm{curl})$ and $u^*\in H_0(\Omega;\mathrm{curl})$ such that $*du=v$ and $*du^*=v^*$. Set $w=u+df+dg$ and $w^*=u^*+df^*$, where f,g and f^* are solutions of the following boundary value problems

$$\begin{cases}-\Delta f &=-\delta u\\ f|_{\partial\Omega} &=0,\end{cases}\qquad \begin{cases}-\Delta g &=0\\ \langle\nu,dg\rangle &=-\langle\nu,u+df\rangle,\end{cases}\qquad \begin{cases}-\Delta f^* &=-\delta u^*\\ f^*|_{\partial\Omega} &=0.\end{cases}$$

Let us make some comments about the setting of these problems. Firstly, note that $\delta u,\delta u^*\in H^{-1}(\Omega)$. Secondly, note that $\delta(u+df)=0$ hence we can ensure that $\langle\nu,u+df\rangle\in B^{-1/2}(\partial\Omega)$. Further, the required compatibility condition for the Neumann's problem is satisfied. Now it is clear that $w\in H(\Omega;\mathrm{curl})\cap\{u\in H_0(\Omega;\mathrm{div}):\delta u=0\}$ and $w^*\in H_0(\Omega;\mathrm{curl})\cap\{u\in H(\Omega;\mathrm{div}):\delta u=0\}$. Furthermore, $\mathbf{curl}\,w=v$ and $\mathbf{curl}^*\,w^*=v^*$. This ends up with the verification of the identities claimed above.

On the other hand, it is an easy computation to check that

$$(\operatorname{Rang}(*d))^\perp=\operatorname{Ker}(*d)^*,\qquad (\operatorname{Rang}(*d)^*)^\perp=\operatorname{Ker}(*d).$$

Hence one can write out the following decompositions of $L^2(\Omega;\Lambda^1 T\mathbb{E})$:

$$\begin{aligned}L^2(\Omega;\Lambda^1 T\mathbb{E})&=\operatorname{Rang}(*d)^*\oplus\operatorname{Ker}(*d)\\ L^2(\Omega;\Lambda^1 T\mathbb{E})&=\operatorname{Rang}(*d)\oplus\operatorname{Ker}(*d)^*.\end{aligned}$$

Note that, for any $u \in \mathrm{Rang}\,(*d)^* \cap H(\Omega;\mathrm{curl}) \cap H_0(\Omega;\mathrm{div})$, one has that

$$\begin{aligned}\inf_{w\in \mathrm{Ker}\,(\mathbf{curl})} \|u+w\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} &\geq \mathrm{dist}_{\|\cdot\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}}(u, \mathrm{Ker}\,(*d)) \\ &= \|R_* u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} = \|u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\,, \end{aligned} \tag{2.28}$$

since $\mathrm{Ker}\,(\mathbf{curl}) \subset \mathrm{Ker}\,(*d)$. Here R_* stands for the orthogonal projector over $\mathrm{Rang}\,(*d)^*$. In the same way, for any $u \in \mathrm{Rang}\,(*d) \cap H_0(\Omega;\mathrm{curl}) \cap H(\Omega;\mathrm{div})$, one has that

$$\begin{aligned}\inf_{w\in \mathrm{Ker}\,(\mathbf{curl}^*)} \|u+w\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} &\geq \mathrm{dist}_{\|\cdot\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}}(u, \mathrm{Ker}\,(*d)^*) \\ &= \|Ru\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})} = \|u\|_{L^2(\Omega;\Lambda^1 T\mathbb{E})}\,, \end{aligned} \tag{2.29}$$

since $\mathrm{Ker}\,(\mathbf{curl}^*) \subset \mathrm{Ker}\,(*d)^*$. Here R stands for the orthogonal projector over $\mathrm{Rang}\,(*d)$.

Observe that, for given $v \in \mathrm{Rang}\,(*d)$ and $v^* \in \mathrm{Rang}\,(*d)^*$, there exist, respectively, unique $u \in \mathrm{Rang}\,(*d)^* \cap H(\Omega;\mathrm{curl}) \cap H_0(\Omega;\mathrm{div})$ and $u^* \in \mathrm{Rang}\,(*d) \cap H_0(\Omega;\mathrm{curl}) \cap H(\Omega;\mathrm{div})$ such that $*du = v$ and $*du^* = v^*$. Thus, from (2.28), (2.29) and the estimates stated in (b) we can conclude that the operators

$$\begin{aligned}(\mathbf{curl})^{-1} &: \mathrm{Rang}\,(*d) \longrightarrow \mathrm{Rang}\,(*d)^* \cap H(\Omega;\mathrm{curl}) \cap H_0(\Omega;\mathrm{div}) \\ (\mathbf{curl}^*)^{-1} &: \mathrm{Rang}\,(*d)^* \longrightarrow \mathrm{Rang}\,(*d) \cap H_0(\Omega;\mathrm{curl}) \cap H(\Omega;\mathrm{div}),\end{aligned}$$

defined as

$$(\mathbf{curl})^{-1}\, v = u, \quad (\mathbf{curl}^*)^{-1}\, v^* = u^*,$$

are $(L^2(\Omega;\Lambda^1 T\mathbb{E}) \to H(\Omega;\mathrm{curl}))$-bounded.

Finally, since $(*d)^{-1} = \iota_{\tan} \circ (\mathbf{curl})^{-1}$ and $((*d)^*)^{-1} = \iota_{\mathrm{nor}} \circ (\mathbf{curl}^*)^{-1}$, the compactness claimed in the statement follows. $\square$

The proof of Theorem 1

Note that in order to prove Theorem 1, it is enough to show –given $J, K \in L^2(\Omega;\Lambda^1 T\mathbb{E})$– the well-posedness of

$$\begin{cases} *dH + i\omega\gamma E = J \\ *dE - i\omega\mu H = K \\ *(\nu \wedge E) = 0, \end{cases} \tag{2.30}$$

for the set of ω's stated in Theorem 1.

Maxwell's equations in (2.30), with non-homogeneous right hand side, can be written as:

$$-MX + C(\omega)X = F,$$

where

$$M = \left(\begin{array}{c|c} & -*d \\ \hline (*d)^* & \end{array}\right), \quad L = \left(\begin{array}{c|c} & ((*d)^*)^{-1} \\ \hline -(*d)^{-1} & \end{array}\right)$$

and

$$X = \left(\frac{E}{H}\right), \quad C(\omega) = i\omega \left(\begin{array}{c|c} \gamma I & \\ \hline & \mu I \end{array}\right), \quad F = \left(\frac{J}{K}\right).$$

Denote $Y = -MX \in \mathrm{Rang}\,(*d) \times \mathrm{Rang}\,(*d)^*$ then

$$X = C(\omega)^{-1}(F - Y).$$

Thus, $Y \in \mathrm{Rang}\,(*d) \times \mathrm{Rang}\,(*d)^*$ satisfies the equation

$$-Y + MC(\omega)^{-1}Y = MC(\omega)^{-1}F.$$

If $P^\perp$ denotes the matrix projector

$$P^\perp = \left(\begin{array}{c|c} R & \\ \hline & R_* \end{array}\right)$$

one has that

$$-LY + P^\perp C(\omega)^{-1}Y = P^\perp C(\omega)^{-1}F.$$

Here R, R_* stand for the orthogonal projectors over $\mathrm{Rang}\,(*d)$ and $\mathrm{Rang}\,(*d)^*$, respectively.

Lemma 2.5 The operator

$$P^\perp C(\omega)^{-1}|_{\mathrm{Rang}\,(*d)\times\mathrm{Rang}\,(*d)^*}$$

can be inverted and its inverse is $(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2$-bounded.

Proof: Consider $Z \in \mathrm{Rang}\,(*d) \times \mathrm{Rang}\,(*d)^*$ with

$$Z = \left(\frac{w^1}{w^2}\right).$$

One has that

$$|\left(P^\perp C(\omega)^{-1}Z \middle| Z\right)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2}| =$$
$$= \left|\int_\Omega \langle R\left(\frac{1}{i\omega\gamma}w^1\right), \overline{w^1}\rangle\, dV + \int_\Omega \langle R_*\left(\frac{1}{i\omega\mu}w^2\right), \overline{w^2}\rangle\, dV\right|$$

$$
\begin{aligned}
&= \left| \int_\Omega \langle \frac{1}{i\omega\gamma} w^1, \overline{w^1} \rangle \, dV + \int_\Omega \langle \frac{1}{i\omega\mu} w^2, \overline{w^2} \rangle \, dV \right| \\
&\geq \frac{1}{|\omega|} \left(\int_\Omega \frac{\varepsilon}{|\gamma|^2} |w^1|^2 \, dV + \int_\Omega \frac{1}{\mu} |w^2|^2 \, dV \right) \\
\geq \frac{C}{|\omega|} \left(\int_\Omega |w^1|^2 \, dV + \int_\Omega |w^2|^2 \, dV \right) &= \frac{C}{|\omega|} \|Z\|^2_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} ,
\end{aligned}
\tag{2.31}
$$

which means that

$$
\frac{C}{|\omega|} \|Z\|_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} \leq \left\| P^\perp C(\omega)^{-1} Z \right\|_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} . \tag{2.32}
$$

With this estimate, the result is a consequence of the Riesz's representation theorem. □

As a direct consequence of the previous lemma we get the following Fredholm-type equation

$$
Y - K(\omega)Y = [P^\perp C(\omega)^{-1}]^{-1} P^\perp C(\omega)^{-1} F,
$$

with

$$
K(\omega) = [P^\perp C(\omega)^{-1}]^{-1} L
$$

compact. Then, by the analytic Fredholm theory (see Section A.2) either $(I - K(\omega))^{-1}$ exists for no $\omega \in \mathbb{C} \setminus \{0\}$, or $(I - K(\omega))^{-1}$ exists for all $\omega \in \mathbb{C} \setminus \{0\}$ except for a discrete subset (i. e. a subset which has no limit points in $\mathbb{C} \setminus \{0\}$). Thus, the next lemma completes the proof of Theorem 1.

Lemma 2.6 $(I - K(\omega))^{-1}$ exists for any $\omega \in \mathbb{C}$ such that $\operatorname{Im}\omega > 0$ or $\operatorname{Im}\omega < -\|\sigma/\varepsilon\|_{L^\infty(\Omega)}$.

Proof: It is an easy computation to verify that, if $Z \in H_0(\Omega; \mathrm{curl}) \times H(\Omega; \mathrm{curl})$ with

$$
Z = \begin{pmatrix} w^1 \\ \overline{w^2} \end{pmatrix},
$$

then

$$
\begin{aligned}
&((-M + C(\omega))Z|Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} + (Z|(-M + C(\omega))Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} = \\
&\quad = -2 \int_\Omega \mu \operatorname{Im}\omega |w^2|^2 \, dV - 2 \int_\Omega (\varepsilon \operatorname{Im}\omega + \sigma) |w^1|^2 \, dV.
\end{aligned}
$$

Observe two simple facts, the first one is that

$$
\begin{aligned}
&|\, ((-M + C(\omega))Z|Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} + (Z|(-M + C(\omega))Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} \,| = \\
&\quad = -2 \int_\Omega \mu \operatorname{Im}\omega |w^2|^2 \, dV - 2 \int_\Omega (\varepsilon \operatorname{Im}\omega + \sigma) |w^1|^2 \, dV,
\end{aligned}
$$

whenever $\mathrm{Im}\,\omega < -\left\|\sigma/\varepsilon\right\|_{L^\infty(\Omega)}$; the second one is that

$$\begin{aligned}
&\left|\,((-M + C(\omega))Z|-Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} + (-Z|(-M + C(\omega))Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2}\,\right| = \\
&\qquad = 2\int_\Omega \mu \mathrm{Im}\,\omega |w^2|^2\, dV + 2\int_\Omega (\varepsilon \mathrm{Im}\,\omega + \sigma)|w^1|^2\, dV,
\end{aligned}$$

whenever $\mathrm{Im}\,\omega > 0$. Hence one gets, for the same set of ω's, the following estimates

$$\|Z\|^2_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} \leq C\left|\,((-M + C(\omega))Z|Z)_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2}\,\right|, \tag{2.33}$$

$$\|Z\|_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2} \leq C\,\|(-M + C(\omega))Z\|_{(L^2(\Omega;\Lambda^1 T\mathbb{E}))^2}\,. \tag{2.34}$$

Once again by the Riesz representation theorem, the statement follows. $\square$

Chapter 3

Stable determination of the electromagnetic coefficients

–Lleva siempre contigo esta ramita de romero –me dijo la gitana antes de curarme el mal de ojos. Así huele el romero que llevo en mi cartera:

> *"La genialidad ayuda pero la perseverancia es más importante. La investigación no es un* sprint*, es una carrera de fondo."*

In this chapter we prove the stable determination of the electromagnetic coefficients stated in Theorem 2. In order to accomplish this task, we proof an estimate relating the boundary data with the electromagnetic properties of the medium. Afterward we construct special solutions to exploit the information coded in the estimate. Finally, we use a Carleman estimate to end up with the estimate of Theorem 2. One of the key points in the construction of special solutions bases on the possibility of transforming Maxwell's equations into a Scrödinger-type equation.

All the computations on this chapter have been done with the approach of vector fields by choosing global euclidean coordinates $\mathcal{E}$. We shall identify $\mathcal{E}(P)$ with $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ for any $P \in \mathbb{E}$. The same could have been done with the language of forms (see [35] and [23]). Nevertheless, we use vector fields because the approach of differential forms requires extra notation and it does not present important advantages for our purposes.

3.1 Maxwell's system as a Schödinger equation

The idea of transforming Maxwell's equations into Schrödinger-type equation was introduced in [33]. This transformation requires extra smoothness of the coefficients, namely $\mu, \gamma \in C^{1,1}(\overline{\Omega})$. The first step in this process is to augment the Maxwell's system

$$\left\{\begin{array}{l} dH + i\omega\gamma *E = 0 \\ dE - i\omega\mu *H = 0. \end{array}\right. \tag{3.1}$$

with two scalar equations:

$$\delta(\gamma E) = 0, \qquad \delta(\mu H) = 0.$$

The information coded in these scalar equations was already present in the initial system, it is enough to take $*d$ in each equation in (3.1).

Next, we introduce a new system inspired in the four mentioned equations. This new system reads as

$$\left\{\begin{array}{l} \delta(\gamma E) + i\omega\gamma\mu h = 0 \\ -\gamma^{-1} d(\gamma e) + *dE - i\omega\mu H = 0 \\ \delta(\mu H) + i\omega\gamma\mu e = 0 \\ \mu^{-1} d(\mu h) + *dH + i\omega\gamma E = 0. \end{array}\right. \tag{3.2}$$

The new terms preserve the physical units of measure of the original four equations. In the euclidean coordinates $\mathcal{E}$, the new system –called henceforth *augmented system*– can be written in vector field notation as it follows

$$\left[\left(\begin{array}{cc|cc} & & & D\cdot \\ & & D & -D\times \\ \hline & D\cdot & & \\ D & D\times & & \end{array}\right) + \left(\begin{array}{cc|cc} \omega\mu & & & D\alpha\cdot \\ & \omega\mu I_3 & D\alpha & \\ \hline & D\beta\cdot & \omega\gamma & \\ D\beta & & & \omega\gamma I_3 \end{array}\right)\right]\left(\begin{array}{c} h \\ H \\ \hline e \\ E \end{array}\right) = 0,$$

where $\alpha = \log\gamma$, $\beta = \log\mu$, I_j is $(j \times j)$-identity matrix, with $j \in \mathbb{N}$ and

$$D\cdot = \frac{1}{i}\left(\begin{array}{ccc} \partial_{x^1} & \partial_{x^2} & \partial_{x^3} \end{array}\right),$$

$$D = \frac{1}{i}\left(\begin{array}{c} \partial_{x^1} \\ \partial_{x^2} \\ \partial_{x^3} \end{array}\right), \quad D\times = \frac{1}{i}\left(\begin{array}{ccc} & -\partial_{x^3} & \partial_{x^2} \\ \partial_{x^3} & & -\partial_{x^1} \\ -\partial_{x^2} & \partial_{x^1} & \end{array}\right).$$

In a much more compact manner we shall express the augmented system as $(P+V)X = 0$, where

$$P = \left(\begin{array}{cc|cc} & & & D\cdot \\ & & D & -D\times \\ \hline & D\cdot & & \\ D & D\times & & \end{array}\right), \qquad V = \left(\begin{array}{cc|cc} \omega\mu & & & D\alpha\cdot \\ & \omega\mu I_3 & D\alpha & \\ \hline & D\beta\cdot & \omega\gamma & \\ D\beta & & & \omega\gamma I_3 \end{array}\right).$$

Note that E, H is a solution for Maxwell's equations, if and only if, $X^t =$

$\begin{pmatrix} h & H^t \,|\, e & E^t \end{pmatrix}$ is a solution for the augmented system and the scalar fields e, h vanish.

The next step is to rescale the augmented system, that is

$$(P+V)\left(\begin{array}{c|c} \mu^{-1/2}I_4 & \\ \hline & \gamma^{-1/2}I_4 \end{array}\right)Y = \left(\begin{array}{c|c} \gamma^{-1/2}I_4 & \\ \hline & \mu^{-1/2}I_4 \end{array}\right)(P+W)Y,$$

where

$$W = \kappa I_8 + \frac{1}{2}\left(\begin{array}{cc|cc} & & & D\alpha\cdot \\ & & D\alpha & D\alpha\times \\ \hline & D\beta\cdot & & \\ D\beta & -D\beta\times & & \end{array}\right), \tag{3.3}$$

with $\kappa = \omega\mu^{1/2}\gamma^{1/2}$. We shall call

$$(P+W)Y = 0$$

the *rescaled system*.

The advantage of rescaling is that

$$0 = (P+W)(P-W^t)Z = (-\Delta+Q)Z, \tag{3.4}$$
$$0 = (P-W^t)(P+W)Z' = (-\Delta+Q')Z', \tag{3.5}$$
$$0 = (P+W^*)(P-\overline{W})\hat{Z} = (-\Delta+\hat{Q})\hat{Z}, \tag{3.6}$$

where $Q, Q', \hat{Q}$ are zeroth-order terms. Here W^t denotes the transposed of W and W^* stands for $\overline{W^t}$. No first order terms appear in (3.4), (3.5) and (3.6), giving as a result a Schrödinger-type equation. Mind

$$Q = -PW^t + WP - WW^t. \tag{3.7}$$

Note that if Z is a solution for (3.4) in Ω, then $Y = (P-W^t)Z$ is a solution for the rescaled system in Ω, hence

$$X = \left(\begin{array}{c|c} \mu^{-1/2}I_4 & \\ \hline & \gamma^{-1/2}I_4 \end{array}\right)Y$$

is a solution for the augmented system. In the same manner, if $\hat{Z}$ is a solution for (3.6), then $\hat{Y} = (P-\overline{W})\hat{Z}$ is a solution for $(P+W^*)\hat{Y} = 0$ in Ω.

For later uses,

$$Q = \frac{1}{2}\left(\begin{array}{cc|cc} \Delta\alpha & & & \\ & 2\nabla^2\alpha - \Delta\alpha I_3 & & \\ \hline & & \Delta\beta & \\ & & & 2\nabla^2\beta - \Delta\beta I_3 \end{array}\right) +$$

$$
-\left(\begin{array}{c|c}
(\kappa^2+\frac{1}{4}(D\alpha\cdot D\alpha))I_4 & \begin{array}{cc} & 2D\kappa\cdot \\ 2D\kappa & \end{array} \\ \hline
\begin{array}{cc} & 2D\kappa\cdot \\ 2D\kappa & \end{array} & (\kappa^2+\frac{1}{4}(D\beta\cdot D\beta))I_4
\end{array}\right), \tag{3.8}
$$

$$
\begin{aligned}
Q' = -\frac{1}{2}&\left(\begin{array}{c|c}
\begin{array}{cc} \Delta\beta & \\ & 2\nabla^2\beta-\Delta\beta I_3 \end{array} & \\ \hline
 & \begin{array}{cc} \Delta\alpha & \\ & 2\nabla^2\alpha-\Delta\alpha I_3 \end{array}
\end{array}\right) \\
&-\left(\begin{array}{c|c}
(\kappa^2+\frac{1}{4}(D\beta\cdot D\beta))I_4 & \begin{array}{cc} & \\ & 2D\kappa\times \end{array} \\ \hline
\begin{array}{cc} & \\ & -2D\kappa\times \end{array} & (\kappa^2+\frac{1}{4}(D\alpha\cdot D\alpha))I_4
\end{array}\right)
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
\hat{Q} = \frac{1}{2}&\left(\begin{array}{c|c}
\begin{array}{cc} -\Delta\beta & \\ & -2\nabla^2\beta+\Delta\beta I_3 \end{array} & \\ \hline
 & \begin{array}{cc} -\Delta\overline{\alpha} & \\ & -2\nabla^2\overline{\alpha}+\Delta\overline{\alpha} I_3 \end{array}
\end{array}\right) \\
&-\left(\begin{array}{c|c}
(\overline{\kappa}^2-\frac{1}{4}(D\beta\cdot D\beta))I_4 & \begin{array}{cc} & \\ & -2D\overline{\kappa}\times \end{array} \\ \hline
\begin{array}{cc} & \\ & 2D\overline{\kappa}\times \end{array} & (\overline{\kappa}^2-\frac{1}{4}(D\overline{\alpha}\cdot D\overline{\alpha}))I_4
\end{array}\right)
\end{aligned} \tag{3.10}
$$

with $\nabla^2 f = (\partial^2_{x_j,x_k} f)^3_{j,k=1}$.

In order to make as concise as possible the presentation of our proofs, we introduce some additional notation. Let Y, Z be in the form

$$
Y = \left(\begin{array}{cc|cc} f^1 & (u^1)^t & f^2 & (u^2)^t \end{array}\right)^t, \qquad Z = \left(\begin{array}{cc|cc} g^1 & (v^1)^t & g^2 & (v^2)^t \end{array}\right)^t,
$$

define

$$
(Y|Z) = \sum_{j=1}^{2}\left(\int_\Omega f^j\overline{g^j}\,dV + \int_\Omega u^j\cdot\overline{v^j}\,dV\right),
$$

$$
(Y|Z)_{\partial\Omega} = \sum_{j=1}^{2}\left(\int_{\partial\Omega} f^j\overline{g^j}\,dA + \int_{\partial\Omega} u^j\cdot\overline{v^j}\,dA\right).
$$

In the first identity we are assuming $f^j, g^j \in C^\infty(\overline{\Omega})$ and $u^j, v^j \in \mathcal{X}\mathbb{E}|_\Omega$ with $j = 1, 2$, while in the second identity $f^j, g^j \in C^\infty(\partial\Omega)$ and $u^j, v^j \in \mathcal{X}\mathbb{E}|_{\partial\Omega}$ with

$j = 1, 2$. The following integration by parts holds

$$(PY|Z) = (P_N Y|Z|_{\partial\Omega})_{\partial\Omega} + (Y|PZ)\,.$$

Here, when A is a (possibly complex) vector field we denote

$$P_A = \frac{1}{i}\left(\begin{array}{cc|cc} & & & A\cdot \\ & & A & -A\times \\ \hline & A\cdot & & \\ A & A\times & & \end{array}\right). \tag{3.11}$$

Finally, for elements Y in the form given above we define, for $|s| > 0$,

$$\|Y\|_{H^s(\Omega;\mathcal{Y})} = \sum_{j=1,2}\left(\|f^j\|_{H^s(\Omega)} + \|u^j\|_{H^s(\Omega;\mathbb{C}^3)}\right),$$

and

$$\|Y\|_{L^2(\Omega;\mathcal{Y})} = \sum_{j=1,2}\left(\|f^j\|_{L^2(\Omega)} + \|u^j\|_{L^2(\Omega;\mathbb{C}^3)}\right).$$

On the other hand, we define, for $0 < |s| < 1$,

$$\|Y\|_{B^s(\partial\Omega;\mathcal{Y})} = \sum_{j=1,2}\left(\|f^j\|_{B^s(\partial\Omega)} + \|u^j\|_{B^s(\partial\Omega;\mathbb{C}^3)}\right),$$

and

$$\|Y\|_{L^2(\partial\Omega;\mathcal{Y})} = \sum_{j=1,2}\left(\|f^j\|_{L^2(\partial\Omega)} + \|u^j\|_{L^2(\partial\Omega;\mathbb{C}^3)}\right).$$

3.2 Relating the boundary data with the coefficients in the interior

Lemma 3.1 Let μ_j, γ_j belong to $C^{0,1}(\overline{\Omega})$. Then one has that, for any Y_1 given by

$$Y_1 = \left(\begin{array}{cc|cc} 0 & \mu_1^{1/2}H_1^t & 0 & \gamma_1^{1/2}E_1^t \end{array}\right)^t$$

with $E_1, H_1 \in H(\Omega;\mathrm{curl})$ solution for (3.1) in Ω with coefficients μ_1, γ_1, and any

$$Y_2 = \left(\begin{array}{cc|cc} f^1 & (u^1)^t & f^2 & (u^2)^t \end{array}\right)^t \in H^1(\Omega) \times H(\Omega;\mathrm{curl}) \times H^1(\Omega) \times H(\Omega;\mathrm{curl})$$

solution for $(P + W_2^*)Y_2 = 0$ in Ω; the following estimate holds:

$$|\,(Y_1|PY_2) - (PY_1|Y_2)\,| \leq$$

$$\leq C\delta_C(C_1,C_2)\left(\left\|\mu_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)}\|g_2\|_{B^{1/2}(\partial\Omega)}+\left\|\gamma_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)}\|z_1\|_{TH(\partial\Omega)}\right.$$
$$\left.+\left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)}\|g_1\|_{B^{1/2}(\partial\Omega)}+\left\|\mu_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)}\|z_2\|_{TH(\partial\Omega)}\right)\|N\times E_1\|_{TH(\partial\Omega)}$$
$$+C\left(\|N\times E_1\|_{TH(\partial\Omega)}+\|N\times H_1\|_{TH(\partial\Omega)}\right)$$
$$\times\left(\left\|\mu_1^{-1/2}-\mu_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)}\|g_2\|_{B^{1/2}(\partial\Omega)}+\left\|\mu_1^{1/2}-\mu_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)}\|z_2\|_{TH(\partial\Omega)}\right.$$
$$\left.+\left\|\gamma_1^{1/2}-\gamma_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)}\|z_1\|_{TH(\partial\Omega)}+\left\|\gamma_1^{-1/2}-\gamma_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)}\|g_1\|_{B^{1/2}(\partial\Omega)}\right).$$

Here $g_1, g_2 \in B^{1/2}(\partial\Omega)$ stand for $g_1 = f^1|_{\partial\Omega}$, $g_2 = f^2|_{\partial\Omega}$ while $z_1, z_2 \in TH(\partial\Omega)$ stand for $z_1 = N\times u^1$, $z_2 = N\times u^2$. Here C_j, with $j = 1, 2$, stands for the Cauchy data set corresponding to μ_j, γ_j. Recall that W_2 is the matrix (3.3) associated to μ_2, γ_2.

Along these notes, μ_j and γ_j should be understood either as themselves or as their traces, according to the context.

Proof: Let L be

$$L=\left(\begin{array}{cc|cc} 0 & \mu_2^{1/2}H_2^t & 0 & \gamma_2^{1/2}E_2^t\end{array}\right)^t,$$

with $E_2, H_2 \in H(\Omega;\mathrm{curl})$ an arbitrary solution for (3.1) with coefficients μ_2, γ_2. Since $(P + W_2^*)Y_2 = 0$ and $(P + W_2)L = 0$, one has that $(L|PY_2) = (PL|Y_2)$, hence

$$(Y_1|PY_2)-(PY_1|Y_2)=(Y_1-L|PY_2)-(P(Y_1-L)|Y_2).$$

On the other hand, we have, using (2.7), (2.18), (2.8) and (2.19), that

$$(Y_1-L|PY_2)-(P(Y_1-L)|Y_2)=$$
$$=i\left\langle N\cdot(\mu_1H_1-\mu_2H_2)\middle|\mu_2^{-1/2}g_2\right\rangle+i\left\langle N\cdot(\mu_1H_1)\middle|(\mu_1^{-1/2}-\mu_2^{-1/2})g_2\right\rangle$$
$$+i\left\langle N\cdot(\gamma_1E_1-\gamma_2E_2)\middle|\overline{\gamma_2^{-1/2}}g_1\right\rangle+i\left\langle N\cdot(\gamma_1E_1)\middle|(\overline{\gamma_1^{-1/2}-\gamma_2^{-1/2}})g_1\right\rangle$$
$$-i\left\langle N\times(H_1-H_2)\middle|N\times(\mu_2^{1/2}z_2)\right\rangle-i\left\langle N\times H_1\middle|N\times((\mu_1^{1/2}-\mu_2^{1/2})z_2)\right\rangle$$
$$+i\left\langle N\times(E_1-E_2)\middle|N\times(\overline{\gamma_2^{1/2}}z_1)\right\rangle+i\left\langle N\times E_1\middle|N\times((\overline{\gamma_1^{1/2}-\gamma_2^{1/2}})z_1)\right\rangle.$$

Furthermore, from the Maxwell's equations one deduces that

$$N\cdot(\gamma_jE_j)=\frac{1}{i\omega}\mathrm{Div}\,(N\times H_j),\quad N\cdot(\mu_jH_j)=-\frac{1}{i\omega}\mathrm{Div}\,(N\times E_j),$$

for $j = 1, 2$. Hence, denoting $N\times E_j = T_j$ and $N\times H_j = S_j$ we obtain

$$
\begin{aligned}
&(Y_1|PY_2) - (PY_1|Y_2) = \\
&= -\frac{1}{\omega}\left\langle \operatorname{Div}(T_1 - T_2)\middle| \mu_2^{-1/2} g_2\right\rangle - \frac{1}{\omega}\left\langle \operatorname{Div} T_1 \middle| (\mu_1^{-1/2} - \mu_2^{-1/2}) g_2\right\rangle \\
&+\frac{1}{\omega}\left\langle \operatorname{Div}(S_1 - S_2)\middle| \overline{\gamma_2^{-1/2} g_1}\right\rangle + \frac{1}{\omega}\left\langle \operatorname{Div} S_1 \middle| \overline{(\gamma_1^{-1/2} - \gamma_2^{-1/2}) g_1}\right\rangle \\
&-i\left\langle S_1 - S_2 \middle| N\times(\mu_2^{1/2} z_2)\right\rangle - i\left\langle S_1 \middle| N\times((\mu_1^{1/2} - \mu_2^{1/2}) z_2)\right\rangle \\
&+i\left\langle T_1 - T_2 \middle| N\times(\overline{\gamma_2^{1/2} z_1})\right\rangle + i\left\langle T_1 \middle| N\times(\overline{(\gamma_1^{1/2} - \gamma_2^{1/2}) z_1})\right\rangle .
\end{aligned}
$$

By using the appropriate dualities and the estimates (2.15), (2.20) and (2.9) we get

$$
\begin{aligned}
&|\,(Y_1|PY_2) - (PY_1|Y_2)\,| \leq \\
&\leq C\left(\|T_1 - T_2\|_{TH(\partial\Omega)} + \|S_1 - S_2\|_{TH(\partial\Omega)}\right)\left(\left\|\mu_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)} \|g_2\|_{B^{1/2}(\partial\Omega)}\right. \\
&+\left\|\gamma_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \|z_1\|_{TH(\partial\Omega)} + \left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)} \|g_1\|_{B^{1/2}(\partial\Omega)} \\
&\left.+\left\|\mu_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \|z_2\|_{TH(\partial\Omega)}\right) + C\left(\left\|\mu_1^{-1/2} - \mu_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)} \|g_2\|_{B^{1/2}(\partial\Omega)}\right. \\
&+\left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \|z_1\|_{TH(\partial\Omega)} + \left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)} \|g_1\|_{B^{1/2}(\partial\Omega)} \\
&\left.+\left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \|z_2\|_{TH(\partial\Omega)}\right)\left(\|T_1\|_{TH(\partial\Omega)} + \|S_1\|_{TH(\partial\Omega)}\right).
\end{aligned}
$$

This estimate holds for all $(T_2, S_2) \in C_2$, since $(E_2\,|\,H_2)^t$ was chosen to be an arbitrary solution for (3.1) with coefficients μ_2, γ_2. Finally, the wanted estimate is a consequence of Definition 1.1. □

Proposition 3.1 Let γ_1, μ_1 and γ_2, μ_2 be in the class B-stable coefficients on the boundary at frequency ω, with B as in Theorem 2. Then, there exists a constant $C(M)$ such that, for any $Z_1 \in H^1(\Omega;\mathcal{Y})$ satisfying $Y_1 = (P - W_1^t)Z_1$ with Y_1 as in Lemma 3.1 and any $Y_2 \in H^1(\Omega;\mathcal{Y})$ as in Lemma 3.1, one has

$$
\begin{aligned}
|\,((Q_1 - Q_2)Z_1|Y_2)\,| &\leq C\,B\big(\delta_C(C_1, C_2)\big)\,\|Z_1\|_{H^1(\Omega;\mathcal{Y})}\,\|Y_2\|_{H^1(\Omega;\mathcal{Y})} \\
&+ C\,B\big(\delta_C(C_1, C_2)\big)\left(\|E_1\|_{H(\Omega;\mathrm{curl})} + \|H_1\|_{H(\Omega;\mathrm{curl})}\right) \\
&\times\left(\left\|f^1\right\|_{H^1(\Omega)} + \left\|u^1\right\|_{H(\Omega;\mathrm{curl})} + \left\|u^2\right\|_{H(\Omega;\mathrm{curl})} + \left\|f^2\right\|_{H^1(\Omega)}\right). \qquad (3.12)
\end{aligned}
$$

Here Q_j is the matrix (3.7) associated to μ_j, γ_j with $j = 1, 2$.

Proof: From (3.7) one has

$$
\begin{aligned}
&((Q_1-Q_2)Z_1|Y_2)=\\
=&-\big(P(W_1^t-W_2^t)Z_1\big|Y_2\big)+((W_1-W_2)PZ_1|Y_2)-\big((W_1W_1^t-W_2W_2^t)Z_1\big|Y_2\big)\\
&=\big((W_1^t-W_2^t)Z_1\big|P_\nu Y_2\big)_{\partial\Omega}-\big(W_1^tZ_1\big|PY_2\big)+\big(W_2^tZ_1\big|PY_2\big)\\
&\quad+\big(W_1(P-W_1^t)Z_1\big|Y_2\big)-(PZ_1|W_2^*Y_2)+\big(W_2^tZ_1\big|W_2^*Y_2\big)\\
&=\big((W_1^t-W_2^t)Z_1\big|P_\nu Y_2\big)_{\partial\Omega}+\big((P-W_1^t)Z_1\big|PY_2\big)+\big(W_1(P-W_1^t)Z_1\big|Y_2\big)\\
&=\big((W_1^t-W_2^t)Z_1\big|P_\nu Y_2\big)_{\partial\Omega}+(Y_1|PY_2)-(PY_1|Y_2)\,.
\end{aligned}
$$

In order to get the penultimate identity, we used twice that $(P+W_2^*)Y_2=0$. In the last one, we used that $Y_1=(P-W_1^t)Z_1$ and that $(P+W_1)Y_1=0$.

It is a straight forward computation to check the next estimate

$$
\begin{aligned}
|\big((W_1^t-W_2^t)Z_1\big|P_\nu Y_2\big)_{\partial\Omega}|\le C\Big(&\|\kappa_1-\kappa_2\|_{L^\infty(\partial\Omega)}+\|\nabla(\beta_1-\beta_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}\\
&+\|\nabla(\alpha_1-\alpha_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}\Big)\,\|Z_1\|_{L^2(\partial\Omega;\mathcal{Y})}\,\|Y_2\|_{L^2(\partial\Omega;\mathcal{Y})}\,.
\end{aligned}
$$

Here, as usually, the norm of $L^\infty(\partial\Omega;\mathbb{C}^3)$ is

$$
\|w\|^2_{L^\infty(\partial\Omega;\mathbb{C}^3)}=\sum_{j=1}^3\big\|w^{(j)}\big\|^2_{L^\infty(\partial\Omega)}\,,
$$

for any $w\in\mathcal{X}(\mathbb{E})|_{\partial\Omega}$.

It is a routine computation to check that, on one hand

$$
\begin{aligned}
\|\kappa_1-\kappa_2\|_{L^\infty(\partial\Omega)}&\le C\Big(\|\gamma_2-\gamma_1\|_{L^\infty(\partial\Omega)}+\|\mu_2-\mu_1\|_{L^\infty(\partial\Omega)}\Big)\\
&\le C\,B\big(\delta_C(C_1,C_2)\big),\\
\|\nabla(\alpha_1-\alpha_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}&\le C\Big(\|\gamma_1-\gamma_2\|_{L^\infty(\partial\Omega)}+\|\nabla(\gamma_1-\gamma_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}\Big)\\
&\le C\,B\big(\delta_C(C_1,C_2)\big),\\
\|\nabla(\beta_1-\beta_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}&\le C\Big(\|\mu_1-\mu_2\|_{L^\infty(\partial\Omega)}+\|\nabla(\mu_1-\mu_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}\Big)\\
&\le C\,B\big(\delta_C(C_1,C_2)\big).
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
&\Big\|\mu_2^{-1/2}\Big\|_{C^{0,1}(\partial\Omega)}+\Big\|\gamma_2^{-1/2}\Big\|_{C^{0,1}(\partial\Omega)}+\Big\|\mu_2^{1/2}\Big\|_{C^{0,1}(\partial\Omega)}+\Big\|\gamma_2^{1/2}\Big\|_{C^{0,1}(\partial\Omega)}\le C\\
&\Big\|\mu_1^{-1/2}-\mu_2^{-1/2}\Big\|_{C^{0,1}(\partial\Omega)}\le C\,\|\mu_1-\mu_2\|_{C^{0,1}(\partial\Omega)}\le C\,B\big(\delta_C(C_1,C_2)\big),
\end{aligned}
$$

$$\left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\partial\Omega)} \leq C\,\|\gamma_1 - \gamma_2\|_{C^{0,1}(\partial\Omega)} \leq C\,B\big(\delta_C(C_1, C_2)\big),$$
$$\left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \leq C\,\|\mu_1 - \mu_2\|_{C^{0,1}(\partial\Omega)} \leq C\,B\big(\delta_C(C_1, C_2)\big),$$
$$\left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\partial\Omega)} \leq C\,\|\gamma_1 - \gamma_2\|_{C^{0,1}(\partial\Omega)} \leq C\,B\big(\delta_C(C_1, C_2)\big).$$

Putting together all these estimates and Lemma 3.1, we get

$$\begin{aligned} |\,((Q_1 - Q_2)Z_1|Y_2)\,| &\leq C\,B\big(\delta_C(C_1, C_2)\big)\,\|Z_1\|_{B^{1/2}(\partial\Omega;\mathcal{Y})}\,\|Y_2\|_{B^{1/2}(\partial\Omega;\mathcal{Y})} \\ &+ C\,B\big(\delta_C(C_1, C_2)\big)\left(\|N\times E_1\|_{TH(\partial\Omega)} + \|N\times H_1\|_{TH(\partial\Omega)}\right) \\ &\times\left(\|g_1\|_{B^{1/2}(\partial\Omega)} + \|z_1\|_{TH(\partial\Omega)} + \|g_2\|_{B^{1/2}(\partial\Omega)} + \|z_2\|_{TH(\partial\Omega)}\right), \end{aligned}$$

hence we deduce the estimate given in the statement. □

3.3 Construction of special solutions

Here we construct two kinds of special solutions, one for the Schrödinger-type equation and another one for $(P + W^*)Y = 0$. The first one was already constructed in [33] but we give here the proof in order to keep track the constants. The second kind of solution is inspired on the solutions given in [23].

Let $B(O;\rho)$ be the open ball centered at the origin O with radius $\rho > 0$ and such that $\Omega \subset B(O;\rho)$. Sometimes $B(O;\rho)$ will be denoted by B to simplify the notation. Let ε_0 and μ_0 denote the electric and magnetic constants, respectively. Extend the coefficients γ, μ defined in Ω to functions in $\mathbb{E}$ –still denoted by γ, μ–, preserving their smoothness and in such a way that $\gamma - \varepsilon_0, \mu - \mu_0$ have compact support in $\overline{B(O;\rho)}$ (regarding to extension see [41]). Note two simple facts. Firstly, the extensions still satisfy the a priori bound and the a priori ellipticity constant in $\mathbb{E}$. Secondly, the extensions of the matrices (3.8), (3.9) (3.10) –still denoted by $Q, Q', \hat{Q}$– satisfy that $\omega^2\varepsilon_0\mu_0 I_8 + Q$, $\omega^2\varepsilon_0\mu_0 I_8 + Q'$ and $\omega^2\varepsilon_0\mu_0 I_8 + \hat{Q}$ have compact support in $\overline{B(O;\rho)}$.

We shall construct solutions for (3.4) in $\mathbb{E}$ with the form of a complex geometrical optic solution (CGO solution for short), that is, in the form

$$Z = e^{i\zeta\cdot x}(L + R),$$

with $L = L(\zeta)$ constant and $\zeta \in \mathbb{C}^3$.

The construction given below base on the following result.

Theorem 3.1 (Sylvester-Uhlmann and Brown) Let G_ζ denote the convolution with the fundamental solution for Fadeev's operator $(-\Delta - 2i\zeta\cdot\nabla)$ with $|\zeta| > 1$.

Then,

$$\|G_\zeta f\|_{L^2_\delta} \leq \frac{C(\delta)}{|\zeta|}\, \|f\|_{L^2_{\delta+1}}\,, \tag{3.13}$$

$$\|\partial_{x^j} G_\zeta f\|_{L^2_\delta} \leq C(\delta)\, \|f\|_{L^2_{\delta+1}}\,, \tag{3.14}$$

for any $f \in L^2_{\delta+1}$ with $-1 < \delta < 0$ and $j = 1, 2, 3$.

The norm in the theorem is

$$\|f\|^2_{L^2_\lambda} = \int_{\mathbb{R}^3} (1 + |x|^2)^\lambda |f|^2\, dx,$$

for $0 < |\lambda| < 1$.

Proof: Estimate (3.13) was proven in [43], while (3.14) was proven in [7]. □

Lemma 3.2 Let δ be a constant such that $-1 < \delta < 0$ and let $\zeta \in \mathbb{C}^3$ be such that $\zeta \cdot \zeta = \omega^2 \varepsilon_0 \mu_0$ with

$$|\zeta| > C(\delta, \rho) \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + Q)^k_j \right\|_{L^\infty(B)}.$$

Then, there exists a

$$Z = e^{i\zeta\cdot x}(L + R)$$

solution for $(-\Delta + Q)Z = 0$ in $\mathbb{E}$, with $Z|_\Omega \in H^2(\Omega; \mathcal{Y})$, $L = L(\zeta)$ constant and $R = R(\zeta)$ satisfying

$$\|R\|_{L^2_\delta \mathcal{Y}} \leq \frac{C(\delta, \rho)}{|\zeta|} |L| \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + Q)^k_j \right\|_{L^\infty(B)}, \tag{3.15}$$

$$\|PR\|_{L^2_\delta \mathcal{Y}} \leq C(\delta, \rho) \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + Q)^k_j \right\|_{L^\infty(B)} \left(|L| + \|R\|_{L^2_\delta \mathcal{Y}} \right). \tag{3.16}$$

Again $\mathcal{Y}$ is meanless, it just stands to remark the form of the elements for which the norms are taken.

Proof: It is an easy matter to check that

$$e^{-i\zeta\cdot x}(-\Delta I_8) e^{i\zeta\cdot x}(L + R) = [-\Delta - 2i\zeta \cdot \nabla + \zeta \cdot \zeta] I_8 (L + R),$$

hence, if $Z = e^{i\zeta\cdot x}(L + R)$ is a solution for (3.4), then R solves

$$((-\Delta - 2i\zeta \cdot \nabla + \omega^2 \varepsilon_0 \mu_0) I_8 + Q) R = -(\omega^2 \varepsilon_0 \mu_0 I_8 + Q) L. \tag{3.17}$$

We will use this equation as the starting point for the construction of the CGO solutions.

Denote $F_\zeta = G_\zeta I_8$. Applying F_ζ to both sides of (3.17) we get

$$(I_8 + F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q))R = -F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q)L. \tag{3.18}$$

On the other hand, we can estimate

$$\left\|F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q)R\right\|_{L^2_\delta\mathcal{Y}} \le \frac{C(\delta,\rho)}{|\zeta|}\sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)^k_j\right\|_{L^\infty(B)}\|R\|_{L^2_\delta\mathcal{Y}},$$

where we applied (3.13) and used the fact that $\omega^2\varepsilon_0\mu_0 I_8+Q$ has compact support in $\overline{B(O;\rho)}$.

Since

$$|\zeta| > C(\delta,\rho)\sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)^k_j\right\|_{L^\infty(B)},$$

the operator $(I_8 + F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q))^{-1}$ is bounded in $L^2_\delta\mathcal{Y}$ and

$$R = -(I_8 + F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q))^{-1}F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q)L,$$

with

$$\|R\|_{L^2_\delta\mathcal{Y}} \le \frac{C(\delta,\rho)}{|\zeta|}|L|\sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)^k_j\right\|_{L^\infty(B)}.$$

Here we used again (3.13) and the fact that $\omega^2\varepsilon_0\mu_0 I_8 + Q$ has compact support in $\overline{B(O;\rho)}$. This compactness is crucial in our arguments.

On the other hand, from (3.18) and (3.14) we deduce that

$$\begin{aligned}\|PR\|_{L^2_\delta\mathcal{Y}} &\le \left\|P[F_\zeta(\omega^2\varepsilon_0\mu_0 I_8 + Q)(R+L)]\right\|_{L^2_\delta\mathcal{Y}}\\ &\le C(\delta,\rho)\sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)^k_j\right\|_{L^\infty(B)}\left(|L| + \|R\|_{L^2_\delta\mathcal{Y}}\right).\end{aligned}$$

□

The arguments given in the above proof can be used to prove existence and uniqueness of solutions for a scalar equation of the same type as (3.17).

We will use the CGO solutions constructed for (3.4) to produce analogous solutions for Maxwell's equations. The procedure follows the ideas exposed in Section 3.1, using the decoupled scalar equations in (3.5).

Proposition 3.2 Let δ be a constant such that $-1 < \delta < 0$ and let $\zeta \in \mathbb{C}^3$ be such that $\zeta\cdot\zeta = \omega^2\varepsilon_0\mu_0$ with

$$|\zeta| > C(\delta,\rho)\left(\sum_{j=1,2}\left\|\omega^2\varepsilon_0\mu_0 + q_j\right\|_{L^\infty(B)} + \sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)^k_j\right\|_{L^\infty(B)}\right),$$

where

$$q_1 = -\frac{1}{2}\Delta\beta - \kappa^2 - \frac{1}{4}(D\beta\cdot D\beta), \quad q_2 = -\frac{1}{2}\Delta\alpha - \kappa^2 - \frac{1}{4}(D\alpha\cdot D\alpha).$$

Then, there exists a

$$Z = e^{i\zeta\cdot x}(L+R)$$

solution of $(-\Delta + Q)Z = 0$ in $\mathbb{E}$, with $Z|_\Omega \in H^2(\Omega;\mathcal{Y})$,

$$L = \frac{1}{|\zeta|}\begin{pmatrix} \zeta\cdot A \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}B \\ \hline \zeta\cdot B \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}A \end{pmatrix},$$

for A, B constant complex vector fields, and R satisfying

$$\|R\|_{L^2_\delta\mathcal{Y}} \leq \frac{C(\delta,\rho)}{|\zeta|}|L|\sum_{j,k=1}^{8}\left\|(\omega^2\varepsilon_0\mu_0 I_8 + Q)_j^k\right\|_{L^\infty(B)},$$

Furthermore, $Y = (P - W^t)Z$ is solution for $(P + W)Y = 0$ in $\mathbb{E}$ and it reads

$$Y = \left(\begin{array}{cc|cc} 0 & \mu^{1/2}H^t & 0 & \gamma^{1/2}E^t \end{array}\right)^t$$

with E, H solution for (3.1) in $\mathbb{E}$.

Proof: Let Y be defined by $Y = (P - W^t)Z$, with Z the solution constructed in Lemma 3.2. If we denote

$$Y = \left(\begin{array}{cc|cc} f^1 & (u^1)^t & f^2 & (u^2)^t \end{array}\right)^t,$$

we will prove that $f^1 = f^2 = 0$.

Note that Y solves (3.5) weakly, with f^j solving the following decoupled equation:

$$(-\Delta + q_j)f^j = 0, \qquad j = 1,2$$

in the weak sense.

Denoting L, R from Lemma 3.2 as

$$L = \left(\begin{array}{cc|cc} l^1 & (L^1)^t & l^2 & (L^2)^t \end{array}\right)^t, \quad R = \left(\begin{array}{cc|cc} r^1 & (R^1)^t & r^2 & (R^2)^t \end{array}\right)^t; \tag{3.19}$$

the functions f^j can be expressed as $f^j = e^{i\zeta\cdot x}(m_j + s_j)$ with

$$m_1 = \zeta\cdot L^2 - \kappa l^1 \qquad s_1 = -\frac{1}{2}D\beta\cdot L^2 + (\zeta + D - \frac{1}{2}D\beta)\cdot R^2 - \kappa r^1,$$

$$m_2 = \zeta \cdot L^1 - \kappa l^2 \qquad s_2 = -\frac{1}{2}D\alpha \cdot L^1 + (\zeta + D - \frac{1}{2}D\alpha) \cdot R^1 - \kappa r^2.$$

It is again a straight forward computation to check that

$$(-\Delta - 2i\zeta \cdot \nabla + \omega^2\varepsilon_0\mu_0 + q_j)(m_j + s_j) = 0, \qquad j = 1, 2. \tag{3.20}$$

Further, using (3.19), (3.15), (3.16) and

$$\operatorname{supp} D\alpha \subset \overline{B(O;\rho)}, \qquad \operatorname{supp} D\beta \subset \overline{B(O;\rho)};$$

one sees that $s_j \in L^2_\delta$. Recall that equation (3.20) has a unique solution in L^2_δ whenever $|\zeta| > C(\delta,\rho)\,\|\omega^2\varepsilon_0\mu_0 + q_j\|_{L^\infty(B)}$, certainly it has to be the trivial one. Therefore, if m_j were also in L^2_δ, then f^j would vanish. Note that in order to have $m_j \in L^2_\delta$, it is enough for its support to be compact. This could be accomplished by choosing l^j, L^j such that

$$\zeta \cdot L^2 = \omega\varepsilon_0^{1/2}\mu_0^{1/2}l^1, \qquad \zeta \cdot L^1 = \omega\varepsilon_0^{1/2}\mu_0^{1/2}l^2.$$

$\square$

Next, we construct the same kind of solutions for the equation $(P+W^*)\hat{Y} = 0$. Since (3.6) holds and $\omega^2\varepsilon_0\mu_0 + \hat{Q}$ has compact support in $\overline{B(O;\rho)}$, the same kind of arguments used in the proof of Lemma 3.2 can be carried out to state the following lemma.

Lemma 3.3 Let δ be a constant such that $-1 < \delta < 0$ and let $\zeta \in \mathbb{C}^3$ be such that $\zeta \cdot \zeta = \omega^2\varepsilon_0\mu_0$ with

$$|\zeta| > C(\delta,\rho) \sum_{j,k=1}^{8} \left\| (\omega^2\varepsilon_0\mu_0 I_8 + \hat{Q})^k_j \right\|_{L^\infty(B)}.$$

Then, there exists a

$$\hat{Z} = e^{i\zeta\cdot x}(\hat{L} + \hat{R})$$

solution of $(-\Delta + \hat{Q})\hat{Z} = 0$ in $\mathbb{E}$, with $\hat{Z}|_\Omega \in H^2(\Omega;\mathcal{Y})$, $\hat{L} = \hat{L}(\zeta)$ constant and $\hat{R} = \hat{R}(\zeta)$ satisfying

$$\left\|\hat{R}\right\|_{L^2_\delta\mathcal{Y}} \le \frac{C(\delta,\rho)}{|\zeta|}|\hat{L}| \sum_{j,k=1}^{8} \left\| (\omega^2\varepsilon_0\mu_0 I_8 + \hat{Q})^k_j \right\|_{L^\infty(B)}$$

$$\left\|P\hat{R}\right\|_{L^2_\delta\mathcal{Y}} \le C(\delta,\rho) \sum_{j,k=1}^{8} \left\| (\omega^2\varepsilon_0\mu_0 I_8 + \hat{Q})^k_j \right\|_{L^\infty(B)} \left(|\hat{L}| + \left\|\hat{R}\right\|_{L^2_\delta\mathcal{Y}}\right).$$

As a consequence of this lemma we get the following proposition.

Proposition 3.3 Let $\zeta \in \mathbb{C}^3$ be such that $\zeta \cdot \zeta = \omega^2 \varepsilon_0 \mu_0$ with

$$|\zeta| > C(\rho) \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q})_j^k \right\|_{L^\infty(B)}.$$

Then, there exists a

$$\hat{Y} = e^{i\zeta \cdot x}(M + S)$$

solution for the equation $(P + W^*)\hat{Y} = 0$ in $\mathbb{E}$, with $\hat{Y}|_\Omega \in H^1(\Omega; \mathcal{Y})$,

$$M = \frac{1}{|\zeta|} \begin{pmatrix} \zeta \cdot \hat{A} \\ -\zeta \times \hat{A} \\ \hline \zeta \cdot \hat{B} \\ \zeta \times \hat{B} \end{pmatrix},$$

for $\hat{A}, \hat{B}$ constant complex vector fields, and S satisfying

$$\|S\|_{L^2(\Omega;\mathcal{Y})} \leq \frac{C(\rho, \Omega)}{|\zeta|} \sum_{j,k=1}^{8} \left(\left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q})_j^k \right\|_{L^\infty(B)} + \|W_j^k\|_{L^\infty(\Omega)} \right).$$

Proof: Let $\hat{Z}$ be the solution constructed in Lemma 3.3, then by equation (3.6), $\hat{Y} = (P - \overline{W})\hat{Z}$ is a solution of $(P + W^*)\hat{Y} = 0$ in $\mathbb{E}$. Considering

$$\hat{L} = \frac{1}{|\zeta|} \left(\begin{array}{cc|cc} 0 & \hat{B}^t & 0 & \hat{A}^t \end{array} \right)^t$$

with $\hat{A}, \hat{B}$ constant complex vector fields, the solution $\hat{Y}$ can be expressed as $\hat{Y} = e^{i\zeta \cdot x}(M + S)$, with

$$M = iP_\zeta \hat{L}, \qquad S = P\hat{R} + iP_\zeta \hat{R} - \overline{W}\hat{L} - \overline{W}\hat{R},$$

where P_ζ is as in (3.11) and S satisfies

$$\begin{aligned}
\|S\|_{L^2(\Omega;\mathcal{Y})} \leq & C(\delta, \rho, \Omega) \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q})_j^k \right\|_{L^\infty(B)} \left(|\hat{L}| + \left\| \hat{R} \right\|_{L^2_\delta \mathcal{Y}} \right) \\
& + C(\delta, \rho, \Omega) |\hat{L}| \sum_{j,k=1}^{8} \left\| (\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q})_j^k \right\|_{L^\infty(B)} \\
& + \sum_{j,k=1}^{8} \|W_j^k\|_{L^\infty(\Omega)} \left(\left\| \hat{L} \right\|_{L^2(\Omega;\mathcal{Y})} + C(\Omega) \left\| \hat{R} \right\|_{L^2_\delta \mathcal{Y}} \right)
\end{aligned}$$

$$\leq \frac{C(\delta,\rho,\Omega)}{|\zeta|}\sum_{j,k=1}^{8}\left(\left\|(\omega^2\varepsilon_0\mu_0 I_8+\hat{Q})_j^k\right\|_{L^\infty(B)}+\left\|W_j^k\right\|_{L^\infty(\Omega)}\right)$$

The last estimate is a consequence of Lemma 3.3. □

3.4 Proof of the log-type estimate

The general ideas of this section go back to [1]. The most relevant difference is the use of a Carleman estimate.

Let μ_1,γ_1 and μ_2,γ_2 be two pairs of coefficients under the hypothesis of Theorem 2 and let us choose

$$\zeta_1 = -\frac{1}{2}\xi + i\left(\tau^2+\frac{|\xi|^2}{4}\right)^{1/2}\eta_1 + \left(\tau^2+\omega^2\varepsilon_0\mu_0\right)^{1/2}\eta_2, \tag{3.21}$$

$$\zeta_2 = \frac{1}{2}\xi - i\left(\tau^2+\frac{|\xi|^2}{4}\right)^{1/2}\eta_1 + \left(\tau^2+\omega^2\varepsilon_0\mu_0\right)^{1/2}\eta_2, \tag{3.22}$$

with $\tau \geq 1$ a free parameter controlling the size of $|\zeta_1|$ and $|\zeta_2|$, where ξ,η_1,η_2 are constant vector fields satisfying $|\eta_1| = |\eta_2| = 1$, $\eta_1\cdot\eta_2 = 0$ and $\eta_j\cdot\xi = 0$ for $j = 1,2$. Note that $\zeta_1 - \overline{\zeta_2} = -\xi$ and

$$\frac{\zeta_1}{|\zeta_1|} = i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} + \mathcal{O}(\tau^{-1}), \qquad \frac{\zeta_2}{|\zeta_2|} = -i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} + \mathcal{O}(\tau^{-1}).$$

Let us consider $Z_1 = e^{i\zeta_1\cdot x}(L_1+R_1), Y_1$ the solutions stated in Proposition 3.2 corresponding to the pair μ_1,γ_1 with $|\zeta_1| > C(\rho,M)$. Recall that

$$L_1 = \frac{1}{|\zeta_1|}\begin{pmatrix} \zeta_1\cdot A_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}B_1 \\ \hline \zeta_1\cdot B_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2}A_1 \end{pmatrix}, \qquad \|R_1\|_{L^2(\Omega;\mathcal{Y})} \leq \frac{C(\rho,\Omega,M)}{|\zeta_1|}.$$

Additionally, consider $Y_2 = e^{i\zeta_2\cdot x}(M_2+S_2)$ the solution stated in Proposition 3.3 corresponding to μ_2,γ_2 with $|\zeta_2| > C(\rho,M)$. Also recall that

$$M_2 = \frac{1}{|\zeta_2|}\begin{pmatrix} \zeta_2\cdot A_2 \\ -\zeta_2\times A_2 \\ \hline \zeta_2\cdot B_2 \\ \zeta_2\times B_2 \end{pmatrix}, \qquad \|S_2\|_{L^2(\Omega;\mathcal{Y})} \leq \frac{C(\rho,\Omega,M)}{|\zeta_2|}.$$

Next we plug these solutions into the estimate (3.12) of Proposition 3.1, with different choices of A_j, B_j.

Choosing $B_1 = B_2 = 0$ and A_1, A_2 such that

$$i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot A_1 = i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot \overline{A_2} = 1$$

one gets, when τ becomes large, that

$$((Q_1 - Q_2)Z_1|Y_2) = \\ = \int_\Omega e^{-i\xi\cdot x}\left(\frac{1}{2}\Delta(\alpha_1 - \alpha_2) + \frac{1}{4}(\nabla\alpha_1\cdot\nabla\alpha_1 - \nabla\alpha_2\cdot\nabla\alpha_2) + (\kappa_2^2 - \kappa_1^2)\right) dV \\ + \mathcal{O}((\tau^2 + |\xi|^2)^{-1/2}),$$

where the implicit constant is $C(\rho, \Omega, M)$. Choosing $A_1 = A_2 = 0$ and B_1, B_2 such that

$$i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot B_1 = i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot \overline{B_2} = 1$$

one gets, when τ becomes large, that

$$((Q_1 - Q_2)Z_1|Y_2) = \\ = \int_\Omega e^{-i\xi\cdot x}\left(\frac{1}{2}\Delta(\beta_1 - \beta_2) + \frac{1}{4}(\nabla\beta_1\cdot\nabla\beta_1 - \nabla\beta_2\cdot\nabla\beta_2) + (\kappa_2^2 - \kappa_1^2)\right) dV \\ + \mathcal{O}((\tau^2 + |\xi|^2)^{-1/2}),$$

where the implicit constant is $C(\rho, \Omega, M)$. Denote

$$f = \mathbf{1}_\Omega\left(\frac{1}{2}\Delta(\alpha_1 - \alpha_2) + \frac{1}{4}(\nabla\alpha_1\cdot\nabla\alpha_1 - \nabla\alpha_2\cdot\nabla\alpha_2) + (\kappa_2^2 - \kappa_1^2)\right)$$
$$g = \mathbf{1}_\Omega\left(\frac{1}{2}\Delta(\beta_1 - \beta_2) + \frac{1}{4}(\nabla\beta_1\cdot\nabla\beta_1 - \nabla\beta_2\cdot\nabla\beta_2) + (\kappa_2^2 - \kappa_1^2)\right),$$

where $\mathbf{1}_\Omega$ is the indicator function of Ω. By Proposition 3.1 and the properties of the special solutions, there exist three constants $c = c(\Omega)$, $C = C(\rho, \Omega, M)$ and $C' = C'(\rho, M)$ such that, for any $\tau \geq C'$ one has

$$|\widehat{f}(\xi)| + |\widehat{g}(\xi)| \leq C\left(B\big(\delta_C(C_1, C_2)\big)e^{c(\tau^2+|\xi|^2)^{1/2}} + (\tau^2 + |\xi|^2)^{-1/2}\right).$$

Note that, for $s_1 < 0$ and $R \geq 1$, one has

$$\|f\|^2_{H^{s_1}(\mathbb{E})} + \|g\|^2_{H^{s_1}(\mathbb{E})} = \int_{|\xi|<R} (1 + |\xi|^2)^{s_1}\left(|\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi \\ + \int_{|\xi|\geq R} (1 + |\xi|^2)^{s_1}\left(|\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2\right) d\xi$$

$$\leq C\left(B\big(\delta_C(C_1,C_2)\big)e^{c(R+\tau)} + \tau^{-1}\right)^2 \int_0^R (1+|r|^2)^{s_1} r^2\,dr$$
$$+ (1+R^2)^{s_1}\left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}\right).$$

Therefore,

$$\|f\|_{H^{s_1}(\mathbb{E})} + \|g\|_{H^{s_1}(\mathbb{E})} \leq C\left(B\big(\delta_C(C_1,C_2)\big)e^{c(R+\tau)} + \tau^{-1}R^{3/2+s_1} + R^{s_1}\right).$$

Now we choose R in such a way that $\tau^{-1}R^{3/2+s_1}$ decays as R^{s_1}, that is, $R = \tau^{2/3}$, hence

$$\|f\|_{H_0^{s_1}(\Omega)} + \|g\|_{H_0^{s_1}(\Omega)} \leq C\left(B\big(\delta_C(C_1,C_2)\big)e^{c\tau} + \tau^{2/3s_1}\right).$$

On the other hand, the a priori bound was chosen to have

$$\|f\|_{H^{s_2}(\Omega)} + \|g\|_{H^{s_2}(\Omega)} \leq C(M),$$

for $0 < s_2 < 1/2$. Finally, by the interpolation estimate (2.6) there exist two constants $C' = C'(\rho, M)$ and $C = C(\rho, \Omega, M, \omega)$ such that, for any $\tau \geq C'$, the following estimate holds

$$\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \leq C\left(B\big(\delta_C(C_1,C_2)\big)e^{c\tau} + \tau^{2/3s_1}\right)^{\theta}, \tag{3.23}$$

with $0 = \theta s_1 + (1-\theta)s_2$.

The idea now is to transfer this estimate from f, g to the difference of the coefficients $\mu_1 - \mu_2$ and $\gamma_1 - \gamma_2$. This can be accomplished by using a Carleman estimate.

A simple computation give:

$$f = \mathbf{1}_\Omega \gamma_1^{-1/2}\left[\Delta(\gamma_1^{1/2} - \gamma_2^{1/2}) + q_f(\gamma_1^{1/2} - \gamma_2^{1/2}) + p_f(\mu_1^{1/2} - \mu_2^{1/2})\right],$$
$$g = \mathbf{1}_\Omega \mu_1^{-1/2}\left[\Delta(\mu_1^{1/2} - \mu_2^{1/2}) + q_g(\mu_1^{1/2} - \mu_2^{1/2}) + p_g(\gamma_1^{1/2} - \gamma_2^{1/2})\right];$$

where

$$q_f = -\left(\frac{\Delta\gamma_2^{1/2}}{\gamma_2^{1/2}} + \omega^2\gamma_1^{1/2}(\gamma_1^{1/2}\mu_1 + \gamma_2^{1/2}\mu_2)\right), \quad p_f = -\omega^2\gamma_1\gamma_2^{1/2}(\mu_1^{1/2} + \mu_2^{1/2}),$$
$$q_g = -\left(\frac{\Delta\mu_2^{1/2}}{\mu_2^{1/2}} + \omega^2\mu_1^{1/2}(\mu_1^{1/2}\gamma_1 + \mu_2^{1/2}\gamma_2)\right), \quad p_g = -\omega^2\mu_1\mu_2^{1/2}(\gamma_1^{1/2} + \gamma_2^{1/2}).$$

Note that, thanks to the a priori bound, we have the following differential inequalities:

$$\begin{aligned}
|\Delta(\gamma_1^{1/2} - \gamma_2^{1/2})| &\le C(M)(|f| + |\gamma_1^{1/2} - \gamma_2^{1/2}| + |\mu_1^{1/2} - \mu_2^{1/2}|), \\
|\Delta(\mu_1^{1/2} - \mu_2^{1/2})| &\le C(M)(|g| + |\gamma_1^{1/2} - \gamma_2^{1/2}| + |\mu_1^{1/2} - \mu_2^{1/2}|).
\end{aligned}$$

In order to simplify the notation, let us write $\phi_1 = \gamma_1^{1/2} - \gamma_2^{1/2}$ and $\phi_2 = \mu_1^{1/2} - \mu_2^{1/2}$. By the differential inequalities written above and the Carleman estimate stated in Appendix B, one has

$$\begin{aligned}
&\sum_{j=1,2} \left(h \left\| e^{\varphi/h}\phi_j \right\|^2_{L^2(\Omega)} + h^3 \left\| e^{\varphi/h}\nabla\phi_j \right\|^2_{L^2(\Omega;\mathbb{C}^3)} \right) \le \\
&\le C'' \sum_{j=1,2} \left(h^4 \left\| e^{\varphi/h}\phi_j \right\|^2_{L^2(\Omega)} + h \left\| e^{\varphi/h}\phi_j \right\|^2_{L^2(\partial\Omega)} + h^3 \left\| e^{\varphi/h}\nabla\phi_j \right\|^2_{L^2(\partial\Omega;\mathbb{C}^3)} \right) \\
&\qquad + C'' h^4 \left(\left\| e^{\varphi/h} f \right\|^2_{L^2(\Omega)} + \left\| e^{\varphi/h} g \right\|^2_{L^2(\Omega)} \right),
\end{aligned}$$

where the constant is $C'' = C''(\Omega, M)$ and $\varphi(x) = 1/2|x - x_0|^2$ with $x_0 \notin \overline{\Omega}$. The terms $h^4 \left\| e^{\varphi/h}\phi_j \right\|^2_{L^2(\Omega)}$, with $j = 1, 2$, can be absorbed by the left hand side of the inequality. Hence, if $d_1 = \inf\{d_e(x; x_0)^2 : x \in \Omega\}$ and $d_2 = \sup\{d_e(x; x_0)^2 : x \in \Omega\}$ we get, for any $h < C''(\Omega, M)^{-1/3}$, that

$$\begin{aligned}
&e^{d_1/h} \sum_{j=1,2} \left(h \|\phi_j\|^2_{L^2(\Omega)} + h^3 \|\nabla\phi_j\|^2_{L^2(\Omega;\mathbb{C}^3)} \right) \le C'' e^{d_2/h} \times \\
&\left[h^4 \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right) + \sum_{j=1,2} \left(h \|\phi_j\|^2_{L^2(\partial\Omega)} + h^3 \|\nabla\phi_j\|^2_{L^2(\partial\Omega;\mathbb{C}^3)} \right) \right].
\end{aligned}$$

But now we can easily estimate

$$\begin{aligned}
\|\phi_1\|_{L^2(\partial\Omega)} &\le C \|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \le CB\big(\delta_C(C_1, C_2)\big), \\
\|\nabla\phi_1\|_{L^2(\partial\Omega;\mathbb{C}^3)} &\le C \left(\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} + \|\nabla(\gamma_1 - \gamma_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)} \right) \\
&\le CB\big(\delta_C(C_1, C_2)\big),
\end{aligned}$$

$$
\begin{aligned}
\|\phi_2\|_{L^2(\partial\Omega)} &\le C\,\|\mu_1-\mu_2\|_{L^\infty(\partial\Omega)} \le CB\big(\delta_C(C_1,C_2)\big),\\
\|\nabla\phi_2\|_{L^2(\partial\Omega;\mathbb{C}^3)} &\le C\left(\|\mu_1-\mu_2\|_{L^\infty(\partial\Omega)} + \|\nabla(\mu_1-\mu_2)\|_{L^\infty(\partial\Omega;\mathbb{C}^3)}\right)\\
&\le CB\big(\delta_C(C_1,C_2)\big),\\
\|\gamma_1-\gamma_2\|_{L^2(\Omega)} + \|\nabla(\gamma_1-\gamma_2)\|_{L^2(\Omega;\mathbb{C}^3)} &\le C\left(\|\phi_1\|_{L^2(\Omega)} + \|\nabla\phi_1\|_{L^2(\Omega;\mathbb{C}^3)}\right),\\
\|\mu_1-\mu_2\|_{L^2(\Omega)} + \|\nabla(\mu_1-\mu_2)\|_{L^2(\Omega;\mathbb{C}^3)} &\le C\left(\|\phi_2\|_{L^2(\Omega)} + \|\nabla\phi_2\|_{L^2(\Omega;\mathbb{C}^3)}\right).
\end{aligned}
$$

The constants above depend on the a priori bounds M. These inequalities and estimate (3.23) gives us

$$
\begin{aligned}
\|\gamma_1-\gamma_2\|_{H^1(\Omega)} + \|\mu_1-\mu_2\|_{H^1(\Omega)} \le & Ce^{\frac{d_2-d_1}{2h}}\left(B\big(\delta_C(C_1,C_2)\big)e^{c\tau} + \tau^{2/3s_1}\right)^{\frac{s_2}{s_2-s_1}}\\
&+ Ce^{\frac{d_2-d_1}{2h}}B\big(\delta_C(C_1,C_2)\big),
\end{aligned}
$$

where $d_2 > d_1$, $s_1 < 0 < s_2 < 1/2$, $c = c(\Omega)$, $C = C(\rho,\Omega,M,)$, $\tau \ge C'(\rho,M)$ and $h < C''(\Omega,M)^{-1/3}$. To end up with the estimate given in the statement, it is enough to note that

$$0 < -\frac{2}{3}\frac{s_1 s_2}{s_2-s_1} < \frac{2}{3}s_2,$$

and to choose the parameter τ as

$$\tau = -\frac{1}{2c}\log B\big(\delta_C(C_1,C_2)\big).$$

Chapter 4

An inverse problem with local boundary data

"El humilde razonamiento de uno vale más que la autoridad de miles."
Galileo Galilei

In this chapter we study the uniqueness and the stability of an inverse boundary value problem with local data. As we have already commented, the basic point in our argument is to construct special solutions vanishing on the unaccessible part of the boundary of our domain U. We do not know how to construct such solutions in general, but assuming certain restrictions on the geometry of U we can do something. If the domain is assumed to be either part of plane or part of a sphere, we can perform a reflection argument that allows us to obtain special solutions with the desired behavior on the boundary. These kind of solutions and the strategy followed in Chapter 3 are the basic ingredients to prove our results.

4.1 About the geometry of U

In order to make precise the geometrical restrictions assumed in Theorem 4, Corollary 5 and Theorem 6, we give the following definitions.

Definition 4.1 We shall say that a bounded Lipschitz domain $U \subset \mathbb{E}$ is *partially flat* if there exists a plane $q \subset \mathbb{E}$ and some euclidean coordinates $\mathcal{E}$ such that,

(i) $q = \{x \in \mathbb{R}^3 : x^3 = 0\}$,

(ii) $U \subset \{x \in \mathbb{R}^3 : x^3 < 0\}$,

(iii) $\Gamma_0 := \mathrm{int}_q(\partial U \cap q) \neq \emptyset$.

We shall say that a bounded Lipschitz domain $U \subset \mathbb{E}$ is *partially spherical* if there exist a point $Q_0 \in \mathbb{E}$, $r_0 > 0$ and some euclidean coordinates $\mathcal{E}$ such that

(i) $Q_0 = y_0$ and $U \subset B(y_0; r_0) := \{y \in \mathbb{R}^3 : |y - y_0| < r_0\}$,

(ii) $\Gamma_0 := \mathrm{int}_{S(y_0;r_0)}(\partial U \cap S(y_0; r_0)) \neq \emptyset$ where $S(y_0; r_0) := \partial B(y_0; r_0)$,

(iii) $0 \in S(y_0; r_0)$ but $0 \notin \overline{U}$.

In the two previous cases, we denote $\Gamma := \partial U \setminus \overline{\Gamma_0}$.

Definition 4.2 We shall say that a partially flat domain U is *suitable* if its *symmetric extension with respect to* q –that is $\Omega := U \cup \Gamma_0 \cup \mathcal{R}(U)$– is also Lipschitz. Here $\mathcal{R}$ denotes the reflection with respect to q and it is defined as $(x^1, x^2, x^3) \longmapsto (x^1, x^2, -x^3)$.

In addition, we shall say that a partially spherical domain U is *suitable* if its *inversion with respect to* $S(0; 2r_0)$ –that is $\Omega := \mathcal{K}(U)$– is a suitable partially flat domain. Here $\mathcal{K}$ denotes the inversion with respect to $S(0; 2r_0)$ and it is defined as $y \longmapsto r_1^2/|y|^2 y$ with $r_1 = 2r_0$.

We have to restrict ourselves to these suitable domains because we need to make an extension of the coefficients preserving their smoothness.

4.2 The domain U is partially flat

Along this section we assume U to be a suitable partially flat domain and we follow the notation in Definition 4.1 and Definition 4.2.

Maxwell's system and the reflection map

Let the coefficients μ, γ be such that $\mu, \gamma \in C^{1,1}(\overline{U})$ with $\partial_{x^3}\mu|_{\Gamma_0} = \partial_{x^3}\gamma|_{\Gamma_0} = 0$ and set $\tilde{\mu}, \tilde{\gamma} : \overline{\Omega} \longrightarrow \mathbb{C}$ two smooth extensions of μ and γ defined as

$$\tilde{\mu}(x^1, x^2, x^3) = \mu(x^1, x^2, -|x^3|), \qquad \tilde{\gamma}(x^1, x^2, x^3) = \gamma(x^1, x^2, -|x^3|),$$

for any $x \in \overline{\Omega}$. Note that the hypothesis $\partial_{x^3}\mu|_{\Gamma_0} = \partial_{x^3}\gamma|_{\Gamma_0} = 0$ allows us to keep the smoothness when extending.

Consider the system

$$\left\{ \begin{array}{l} \nabla \times H + i\omega\tilde{\gamma}E = 0 \\ \nabla \times E - i\omega\tilde{\mu}H = 0 \end{array} \right. \tag{4.1}$$

in Ω. The push-forward of the reflection map $\mathcal{R}$ reads

$$\mathcal{R}_* = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Let f be a smooth function and u, u' two vector fields on $\mathbb{E}$. Let g, v, v' denote the function and the vector fields given by

$$g(x) := f(\mathcal{R}(x)), \qquad v_x := \mathcal{R}_* u_{\mathcal{R}(x)}, \qquad v'_x := \mathcal{R}_* u_{\mathcal{R}(x)}.$$

It is an straight forward computation to check that

$$(\nabla \cdot v)(x) = (\nabla \cdot u)(\mathcal{R}(x)), \tag{4.2}$$

$$(\nabla g)_x = \mathcal{R}_* (\nabla f)_{\mathcal{R}(x)}, \tag{4.3}$$

$$(\nabla \times v)_x = -\mathcal{R}_* (\nabla \times u)_{\mathcal{R}(x)}, \tag{4.4}$$

$$(v \times v')_x = -\mathcal{R}_* (u \times u')_{\mathcal{R}(x)}. \tag{4.5}$$

On the other hand, let a be a smooth function defined in $\{x \in \mathbb{R}^3 : x^3 < 0\}$ and set $\tilde{a}$, the extension of a to $\mathbb{E}$, defined as $\tilde{a}(x^1, x^2, x^3) = a(x^1, x^2, -|x^3|)$. Then

$$(\nabla \tilde{a})_x = \mathcal{R}_* (\nabla \tilde{a})_{\mathcal{R}(x)}. \tag{4.6}$$

Lemma 4.1 Given

$$Y = \left(\begin{array}{cc|cc} 0 & \tilde{\mu}^{1/2} H^t & 0 & \tilde{\gamma}^{1/2} E^t \end{array} \right)^t,$$

such that $E, H \in H(\Omega; \mathrm{curl})$ is a solution of (4.1) in Ω, one has that $E - \dot{E}, H - \dot{H}$, with

$$\dot{E}_x := \mathcal{R}_* E_{\mathcal{R}(x)}, \qquad \dot{H}_x := -\mathcal{R}_* H_{\mathcal{R}(x)};$$

is also a solution of (4.1) in Ω satisfying

$$N \times (E - \dot{E})|_{\Gamma_0} = 0. \tag{4.7}$$

Proof: Let E, H be a solution of (4.1) in Ω. It is an immediate consequence of (4.4) and the definition of $\tilde{\mu}, \tilde{\gamma}$ in Ω that $\dot{E}, \dot{H}$ is also a solution for (4.1) in Ω. Further, from the weak definition of tangential trace one can derive that $N \times (E - \dot{E})|_{\Gamma_0} = 0$. Indeed, let $w \in B^{1/2}(\partial U; \mathbb{C}^3)$ such that $\operatorname{supp} w \subset \overline{\Gamma_0}$ and consider $v \in H^1(U; \mathbb{C}^3)$ such that $v|_{\partial U} = w$, then

$$\begin{aligned}
\left\langle N \times E - N \times \dot{E} \middle| w \right\rangle_{\partial U} &= \int_U (\nabla \times E - \nabla \times \dot{E}) \cdot \overline{v} \, dV - \int_U (E - \dot{E}) \cdot \overline{\nabla \times v} \, dV \\
&= \int_U \nabla \times E \cdot \overline{v} - E \cdot \overline{\nabla \times v} \, dV + \int_U (\nabla \times E)_{\mathcal{R}} \cdot \overline{\mathcal{R}_* v} + E_{\mathcal{R}} \cdot \overline{\mathcal{R}_* \nabla \times v} \, dV \\
&= \int_U \nabla \times E \cdot \overline{v} - E \cdot \overline{\nabla \times v} \, dV + \int_{\mathcal{R}(U)} (\nabla \times E) \cdot \overline{\mathcal{R}_* v_{\mathcal{R}}} - E \cdot \overline{(\nabla \times \mathcal{R}_* v_{\mathcal{R}})} \, dV \\
&= \int_\Omega \nabla \times E \cdot \overline{u} - E \cdot \overline{\nabla \times u} \, dV = \langle N \times E | u \rangle_{\partial \Omega} = 0.
\end{aligned}$$

Here we have used (4.4) twice, and the fact that u, defined as v in U and as $\mathcal{R}_* v_{\mathcal{R}}$ in $\mathcal{R}(U)$, belongs to $H^1(\Omega;\mathbb{C}^3)$ and $u|_{\partial\Omega} = 0$. $\square$

Lemma 4.2 Given

$$Y = \left(\begin{array}{cc|cc} f^1 & u^1 & f^2 & u^2 \end{array} \right)$$

solution of $(P + W^*)Y = 0$ in Ω with $f^j \in H^1(\Omega)$ and $u^j \in H(\Omega;\mathrm{curl})$, one has that $Y - \dot{Y}$ is also a solution of $(P + W^*)(Y - \dot{Y}) = 0$ in Ω. Here W denotes the matrix (3.3) for coefficients $\tilde{\mu}, \tilde{\gamma}$ and $\dot{Y}_x := \dot{J} Y_{\mathcal{R}(x)}$ with

$$Y_x = \left(\begin{array}{cc|cc} f^1(x) & u^1_x & f^2(x) & u^2_x \end{array} \right) \qquad \dot{J} := \left(\begin{array}{cc|cc} 1 & & & \\ & -\mathcal{R}_* & & \\ \hline & & -1 & \\ & & & \mathcal{R}_* \end{array} \right).$$

Additionally,

$$(f^1 - \dot{f}^1)|_{\Gamma_0} = 0, \qquad N \times (u^2 - \dot{u}^2)|_{\Gamma_0} = 0. \tag{4.8}$$

Proof: The first part of the lemma follows from

$$(P\dot{Y})_x = \dot{J}(PY)_{\mathcal{R}(x)} \tag{4.9}$$

and

$$(W^*\dot{Y})_x = \dot{J}(W^*Y)_{\mathcal{R}(x)}. \tag{4.10}$$

The identity (4.9) is a consequence of (4.2), (4.3) and (4.4). The identity (4.10) follows from (4.6) and (4.5).

Additionally, $(f^1 - \dot{f}^1)|_{\Gamma_0} = 0$ since $f^1 \in H^1(\Omega)$ and $N \times (u^2 - \dot{u}^2)|_{\Gamma_0} = 0$ as we showed in the proof of Lemma 4.1. $\square$

Relating the local boundary data with the interior

Lemma 4.3 Let μ_j, γ_j belong to $C^{0,1}(\overline{U})$. Then, for any Y_1 given as in the hypothesis of Lemma 4.1 with coefficients $\tilde{\mu}_1, \tilde{\gamma}_1$ and any Y_2 given as in the hypothesis of Lemma 4.2 with coefficients $\tilde{\mu}_2, \tilde{\gamma}_2$, one has that the elements $\mathcal{E}_1 = E_1 - \dot{E}_1$, $\mathcal{H}_1 = H_1 - \dot{H}_1$ and $\mathcal{Y}_2 = Y_2 - \dot{Y}_2$, expressed in the form

$$Y_1 = \left(\begin{array}{cc|cc} 0 & \tilde{\mu}_1^{1/2} H_1^t & 0 & \tilde{\gamma}_1^{1/2} E_1^t \end{array} \right)^t \quad \mathcal{Y}_2 = \left(\begin{array}{cc|cc} f^1 & (u^1)^t & f^2 & (u^2)^t \end{array} \right)^t,$$

satisfy the following estimate

$$|\,(Y_1|P\mathcal{Y}_2)_\Omega - (PY_1|\mathcal{Y}_2)_\Omega\,| \leq$$

$$\leq C\delta_C(C^1_\Gamma, C^2_\Gamma) \left(\left\| \mu_2^{-1/2} \right\|_{C^{0,1}(\overline{\Gamma})} \left\| g_2|_\Gamma \right\|_{B^{1/2}(\Gamma)} + \left\| \gamma_2^{1/2} \right\|_{C^{0,1}(\overline{\Gamma})} \left\| z_1|_\Gamma \right\|_{TH(\Gamma)} + \right.$$

$$+\left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|g_1\right\|_{B_0^{1/2}(\Gamma)}+\left\|\mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|z_2\right\|_{TH_0(\Gamma)}\Big)\left\|N\times\mathcal{E}_1\right\|_{TH_0(\Gamma)}$$
$$+C\left(\left\|\mu_1^{-1/2}-\mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|g_2|_\Gamma\right\|_{B^{1/2}(\Gamma)}\right.$$
$$+\left\|\gamma_1^{1/2}-\gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|z_1|_\Gamma\right\|_{TH(\Gamma)}+\left\|\gamma_1^{-1/2}-\gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|g_1\right\|_{B_0^{1/2}(\Gamma)}$$
$$\left.+\left\|\mu_1^{1/2}-\mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|z_2\right\|_{TH_0(\Gamma)}\right)\left(\left\|N\times\mathcal{E}_1\right\|_{TH_0(\Gamma)}+\left\|N\times\mathcal{H}_1|_\Gamma\right\|_{TH(\Gamma)}\right).$$

Here $g_1, g_2 \in B^{1/2}(\partial U)$ stand for $g_1 = f^1|_{\partial U}$, $g_2 = f^2|_{\partial U}$ and $z_1, z_2 \in TH(\partial U)$ stand for $z_1 = N\times u^1$, $z_2 = N\times u^2$. Here $C_\Gamma^j = C(\mu_j, \gamma_j; \Gamma)$ with $j = 1, 2$.

Proof: It is easy to check, using (4.9) that

$$(Y_1|PY_2)_\Omega - (PY_1|Y_2)_\Omega = (\mathcal{Y}_1|P\mathcal{Y}_2)_U - (P\mathcal{Y}_1|\mathcal{Y}_2)_U\,,$$

where $\mathcal{Y}_1 = Y_1 - \dot{Y}_1$. Let $\mathcal{L}$ be

$$\mathcal{L} = \left(\begin{array}{cc|cc} 0 & \mu_2^{1/2}\mathcal{H}_2^t & 0 & \gamma_2^{1/2}\mathcal{E}_2^t \end{array}\right)^t,$$

with $\mathcal{E}_2, \mathcal{H}_2 \in H(U;\mathrm{curl})$ an arbitrary solution of

$$\nabla\times\mathcal{H}_2 + i\omega\gamma_2\mathcal{E}_2 = 0, \qquad \nabla\times\mathcal{E}_2 - i\omega\mu_2\mathcal{H}_2 = 0 \tag{4.11}$$

in U and satisfying $\operatorname{supp} N\times\mathcal{E}_2 \subset \overline{\Gamma}$. Since $(P + W_2^*)\mathcal{Y}_2 = 0$ and $(P + W_2)\mathcal{L} = 0$ in U, one has that $(\mathcal{L}|P\mathcal{Y}_2)_U = (P\mathcal{L}|\mathcal{Y}_2)_U$, hence

$$(\mathcal{Y}_1|P\mathcal{Y}_2)_U - (P\mathcal{Y}_1|\mathcal{Y}_2)_U = (\mathcal{Y}_1 - \mathcal{L}|P\mathcal{Y}_2)_U - (P(\mathcal{Y}_1 - \mathcal{L})|\mathcal{Y}_2)_U\,.$$

On the other hand, we have, using (2.7), (2.18), (2.8) and (2.19) , that

$$(\mathcal{Y}_1 - \mathcal{L}|P\mathcal{Y}_2)_U - (P(\mathcal{Y}_1 - \mathcal{L})|\mathcal{Y}_2)_U =$$
$$= i\left\langle N\cdot(\mu_1\mathcal{H}_1 - \mu_2\mathcal{H}_2)\middle|\mu_2^{-1/2}g_2\right\rangle + i\left\langle N\cdot(\mu_1\mathcal{H}_1)\middle|(\mu_1^{-1/2} - \mu_2^{-1/2})g_2\right\rangle$$
$$+i\left\langle N\cdot(\gamma_1\mathcal{E}_1 - \gamma_2\mathcal{E}_2)\middle|\overline{\gamma_2^{-1/2}}g_1\right\rangle + i\left\langle N\cdot(\gamma_1\mathcal{E}_1)\middle|\overline{(\gamma_1^{-1/2} - \gamma_2^{-1/2})}g_1\right\rangle$$
$$-i\left\langle N\times(\mathcal{H}_1 - \mathcal{H}_2)\middle|N\times(\mu_2^{1/2}z_2)\right\rangle - i\left\langle N\times\mathcal{H}_1\middle|N\times((\mu_1^{1/2} - \mu_2^{1/2})z_2)\right\rangle$$
$$-i\left\langle N\times(\gamma_2^{1/2}N\times(\mathcal{E}_1 - \mathcal{E}_2))\middle|z_1\right\rangle - i\left\langle N\times((\gamma_1^{1/2} - \gamma_2^{1/2})N\times\mathcal{E}_1)\middle|z_1\right\rangle.$$

Furthermore, from the Maxwell's equations one deduces that

$$N\cdot(\gamma_j\mathcal{E}_j) = \frac{1}{i\omega}\mathrm{Div}\,(N\times\mathcal{H}_j), \quad N\cdot(\mu_j\mathcal{H}_j) = -\frac{1}{i\omega}\mathrm{Div}\,(N\times\mathcal{E}_j),$$

for $j = 1, 2$. Hence,

$$
\begin{aligned}
&(\mathcal{Y}_1|P\mathcal{Y}_2)_U - (P\mathcal{Y}_1|\mathcal{Y}_2)_U = \\
&= -\frac{1}{\omega}\left\langle \operatorname{Div}(N\times\mathcal{E}_1 - N\times\mathcal{E}_2)\middle| \mu_2^{-1/2}\mathcal{g}_2\right\rangle - \frac{1}{\omega}\left\langle \operatorname{Div} N\times\mathcal{E}_1 \middle| (\mu_1^{-1/2} - \mu_2^{-1/2})\mathcal{g}_2\right\rangle \\
&+\frac{1}{\omega}\left\langle \operatorname{Div}(N\times\mathcal{H}_1 - N\times\mathcal{H}_2)\middle| \overline{\gamma_2^{-1/2}}\mathcal{g}_1\right\rangle + \frac{1}{\omega}\left\langle \operatorname{Div} N\times\mathcal{H}_1 \middle| \overline{(\gamma_1^{-1/2} - \gamma_2^{-1/2})}\mathcal{g}_1\right\rangle \\
&-i\left\langle N\times\mathcal{H}_1 - N\times\mathcal{H}_2 \middle| N\times(\mu_2^{1/2}\mathcal{z}_2)\right\rangle - i\left\langle N\times\mathcal{H}_1 \middle| N\times((\mu_1^{1/2} - \mu_2^{1/2})\mathcal{z}_2)\right\rangle \\
&-i\left\langle N\times(\gamma_2^{1/2}(N\times\mathcal{E}_1 - N\times\mathcal{E}_2))\middle| \mathcal{z}_1\right\rangle - i\left\langle N\times((\gamma_1^{1/2} - \gamma_2^{1/2})N\times\mathcal{E}_1)\middle| \mathcal{z}_1\right\rangle .
\end{aligned}
$$

Let us denote $N\times\mathcal{E}_j = T_j$, $N\times\mathcal{H}_j|_\Gamma = S_j$, then by using the appropriate dualities, the boundary conditions (4.7), (4.8) and the estimates (2.16), (2.17), (2.21), (2.22) and (2.23) we get

$$
\begin{aligned}
&|\,(Y_1|P\mathcal{Y}_2)_\Omega - (PY_1|\mathcal{Y}_2)_\Omega\,| \leq \\
&\leq C\left(\|T_1 - T_2\|_{TH_0(\Gamma)} + \|S_1 - S_2\|_{TH(\Gamma)}\right)\left(\left\|\mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|\mathcal{g}_2|_\Gamma\right\|_{B^{1/2}(\Gamma)}\right. \\
&+\left\|\gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{z}_1|_\Gamma\|_{TH(\Gamma)} + \left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{g}_1\|_{B_0^{1/2}(\Gamma)} \\
&\left.+\left\|\mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{z}_2\|_{TH_0(\Gamma)}\right) + C\left(\left\|\mu_1^{-1/2} - \mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\left\|\mathcal{g}_2|_\Gamma\right\|_{B^{1/2}(\Gamma)}\right. \\
&+\left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{z}_1|_\Gamma\|_{TH(\Gamma)} + \left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{g}_1\|_{B_0^{1/2}(\Gamma)} \\
&\left.+\left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})}\|\mathcal{z}_2\|_{TH_0(\Gamma)}\right)\left(\|T_1\|_{TH_0(\Gamma)} + \|S_1\|_{TH(\Gamma)}\right).
\end{aligned}
$$

This estimate holds for all $(T_2, S_2) \in C_\Gamma^2$, since $\mathcal{E}_2, \mathcal{H}_2$ was chosen to be an arbitrary solution of (4.11) in U satisfying $\operatorname{supp} N\times\mathcal{E}_2 \subset \overline{\Gamma}$. Finally, the wanted estimate is a consequence of Definition 1.4. □

Proposition 4.1 Let γ_1, μ_1 and γ_2, μ_2 be in the class of B-stable coefficients on Γ at frequency ω. Then, there exists a constant $C(M)$ such that, for any $Z_1 \in H^1(\Omega; \mathcal{Y})$ satisfying $Y_1 = (P - W_1^t)Z_1$ with Y_1 as in Lemma 4.3 and any $\mathcal{Y}_2 \in H^1(\Omega; \mathcal{Y})$ as in Lemma 4.3, one has

$$
\begin{aligned}
|\,((Q_1 - Q_2)Z_1|\mathcal{Y}_2)_\Omega\,| &\leq C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big)\,\|Z_1\|_{H^1(\Omega;\mathcal{Y})}\,\|\mathcal{Y}_2\|_{H^1(\Omega;\mathcal{Y})} \\
&+ C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big)\left(\|\mathcal{E}_1\|_{H(U;\mathrm{curl})} + \|\mathcal{H}_1\|_{H(U;\mathrm{curl})}\right) \\
&\times\left(\left\|f^1\right\|_{H^1(U)} + \left\|u^1\right\|_{H(U;\mathrm{curl})} + \left\|f^2\right\|_{H^1(U)} + \left\|u^2\right\|_{H(U;\mathrm{curl})}\right),
\end{aligned}
$$

Here Q_j is the matrix (3.7) associates to $\tilde{\mu}_j, \tilde{\gamma}_j$ with $j = 1, 2$.

Proof: From (3.7) one has

$$\begin{aligned}
((Q_1 - Q_2)Z_1|\mathcal{Y}_2)_\Omega &= -\left(P(W_1^t - W_2^t)Z_1\middle|\mathcal{Y}_2\right)_\Omega \\
&+ ((W_1 - W_2)PZ_1|\mathcal{Y}_2)_\Omega - \left((W_1W_1^t - W_2W_2^t)Z_1\middle|\mathcal{Y}_2\right)_\Omega \\
&= \left((W_1^t - W_2^t)Z_1\middle|P_\nu\mathcal{Y}_2\right)_{\partial\Omega} - \left(W_1^tZ_1\middle|P\mathcal{Y}_2\right)_\Omega + \left(W_2^tZ_1\middle|P\mathcal{Y}_2\right)_\Omega \\
&+ \left(W_1(P - W_1^t)Z_1\middle|\mathcal{Y}_2\right)_\Omega - (PZ_1|W_2^*\mathcal{Y}_2)_\Omega + \left(W_2^tZ_1\middle|W_2^*\mathcal{Y}_2\right)_\Omega \\
&= \left((W_1^t - W_2^t)Z_1\middle|P_\nu\mathcal{Y}_2\right)_{\partial\Omega} + \left((P - W_1^t)Z_1\middle|P\mathcal{Y}_2\right)_\Omega + \left(W_1(P - W_1^t)Z_1\middle|\mathcal{Y}_2\right)_\Omega \\
&= \left((W_1^t - W_2^t)Z_1\middle|P_\nu\mathcal{Y}_2\right)_{\partial\Omega} + (Y_1|P\mathcal{Y}_2)_\Omega - (PY_1|\mathcal{Y}_2)_\Omega\,.
\end{aligned}$$

In order to get the penultimate identity, we used twice that $(P + W_2^*)\mathcal{Y}_2 = 0$, while to get the last one, we used that $Y_1 = (P - W_1^t)Z_1$ and that $(P + W_1)Y_1 = 0$.

It is a straight forward computation to check the next estimate

$$\begin{aligned}
\left|\left((W_1^t - W_2^t)Z_1\middle|P_\nu Y_2\right)_{\partial\Omega}\right| \le C\Big(&\|\kappa_1 - \kappa_2\|_{L^\infty(\Gamma)} + \|\nabla(\beta_1 - \beta_2)\|_{L^\infty(\Gamma;\mathbb{C}^3)} \\
&+ \|\nabla(\alpha_1 - \alpha_2)\|_{L^\infty(\Gamma;\mathbb{C}^3)}\Big)\,\|Z_1\|_{L^2(\partial\Omega;\mathcal{Y})}\,\|\mathcal{Y}_2\|_{L^2(\partial\Omega;\mathcal{Y})}\,.
\end{aligned}$$

Here, as usually, the norm of $L^\infty(\Gamma;\mathbb{C}^3)$ is

$$\|w\|^2_{L^\infty(\Gamma;\mathbb{C}^3)} = \sum_{j=1}^{3} \left\|w^{(j)}\right\|^2_{L^\infty(\Gamma)}\,,$$

for any vector field w. It is a routine computation to check that, on one hand

$$\begin{aligned}
\|\kappa_1 - \kappa_2\|_{L^\infty(\Gamma)} &\le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big),\\
\|\nabla(\alpha_1 - \alpha_2)\|_{L^\infty(\Gamma;\mathbb{C}^3)} &\le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big),\\
\|\nabla(\beta_1 - \beta_2)\|_{L^\infty(\Gamma;\mathbb{C}^3)} &\le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big).
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\left\|\mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} + \left\|\gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} + \left\|\mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} + \left\|\gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} &\le C\\
\left\|\mu_1^{-1/2} - \mu_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} + \left\|\mu_1^{1/2} - \mu_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} &\le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big),\\
\left\|\gamma_1^{-1/2} - \gamma_2^{-1/2}\right\|_{C^{0,1}(\overline{\Gamma})} + \left\|\gamma_1^{1/2} - \gamma_2^{1/2}\right\|_{C^{0,1}(\overline{\Gamma})} &\le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big),
\end{aligned}$$

With all these estimates and Lemma 4.3 in mind, we get

$$\left|((Q_1 - Q_2)Z_1|\mathcal{Y}_2)_\Omega\right| \le C\,B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big)\,\|Z_1\|_{B^{1/2}(\partial\Omega;\mathcal{Y})}\,\|\mathcal{Y}_2\|_{B^{1/2}(\partial\Omega;\mathcal{Y})} +$$

$$+\,C\,B\big(\delta_C(C^1_\Gamma,C^2_\Gamma)\big)\left(\|N\times\mathcal{E}_1\|_{TH_0(\Gamma)}+\|N\times\mathcal{H}_1|_\Gamma\|_{TH(\Gamma)}\right)$$
$$\times\left(\|\mathcal{g}_1\|_{B^{1/2}_0(\Gamma)}+\|z_1|_\Gamma\|_{TH(\Gamma)}+\|\mathcal{g}_2|_\Gamma\|_{B^{1/2}(\Gamma)}+\|z_2\|_{TH_0(\Gamma)}\right),$$

hence we deduce the estimate given in the statement. □

Proof of the stability

Let μ_1,γ_1 and μ_2,γ_2 be two pairs of coefficients under the hypothesis of Theorem 4. Consider $\tilde{\mu}_j,\tilde{\gamma}_j$ with $j=1,2$ their even extensions to Ω. Let $B(O;\rho)$ be the open ball centered at the origin O with radius $\rho>0$ and such that $\Omega\subset B(O;\rho)$. Sometimes $B(O;\rho)$ will be denoted by B to simplify the notation. Let ε_0 and μ_0 denote the electric and magnetic constants, respectively. Extend the coefficients $\tilde{\mu}_j,\tilde{\gamma}_j$ defined in Ω to functions in $\mathbb{E}$ –still denoted by $\tilde{\mu}_j,\tilde{\gamma}_j$–, preserving their smoothness and in such a way that $\tilde{\mu}_j-\mu_0,\tilde{\gamma}_j-\varepsilon_0$ have compact support in $\overline{B(O;\rho)}$ (regarding to extension see [41]). Note two simple facts. Firstly, the extensions still satisfy the a priori bound and the a priori ellipticity constant in $\mathbb{E}$. Secondly, the extensions of the matrices (3.8), (3.9) (3.10) –still denoted by $Q_j,Q'_j,\hat{Q}_j$– satisfy that $\omega^2\varepsilon_0\mu_0I_8+Q_j$, $\omega^2\varepsilon_0\mu_0I_8+Q'_j$ and $\omega^2\varepsilon_0\mu_0I_8+\hat{Q}_j$ have compact support in $\overline{B(O;\rho)}$.

Choose

$$\zeta_1=-\frac{1}{2}\xi+i\left(\tau^2+\frac{|\xi|^2}{4}\right)^{1/2}\eta_1+\left(\tau^2+\omega^2\varepsilon_0\mu_0\right)^{1/2}\eta_2, \tag{4.12}$$
$$\zeta_2=\frac{1}{2}\xi-i\left(\tau^2+\frac{|\xi|^2}{4}\right)^{1/2}\eta_1+\left(\tau^2+\omega^2\varepsilon_0\mu_0\right)^{1/2}\eta_2, \tag{4.13}$$

with $\tau\geq 1$ a free parameter controlling the size of $|\zeta_1|$ and $|\zeta_2|$, where ξ,η_1,η_2 constant vector fields satisfying $|\eta_1|=|\eta_2|=1$, $\eta_1\cdot\eta_2=0$, $\eta_j\cdot\xi=0$ for $j=1,2$ and $\xi\neq e_3$. More precisely, if ξ reads in the coordinates $\mathcal{E}$ as

$$\xi=\begin{pmatrix}\xi^{(1)} & \xi^{(2)} & \xi^{(3)}\end{pmatrix}^t,$$

we choose

$$\eta_1=\frac{1}{|\xi'|}\begin{pmatrix}\xi^{(2)}\\-\xi^{(1)}\\0\end{pmatrix}\qquad \eta_2=\eta_1\times\frac{\xi}{|\xi|}=\frac{1}{|\xi'||\xi|}\begin{pmatrix}-\xi^{(1)}\xi^{(3)}\\-\xi^{(2)}\xi^{(3)}\\|\xi'|^2\end{pmatrix},$$

with $|\xi'|^2=(\xi^{(1)})^2+(\xi^{(2)})^2$. Observe that $\zeta_1-\overline{\zeta_2}=-\xi$ and

$$\frac{\zeta_1}{|\zeta_1|}=i\frac{\eta_1}{\sqrt{2}}+\frac{\eta_2}{\sqrt{2}}+\mathcal{O}(\tau^{-1}),\qquad \frac{\zeta_2}{|\zeta_2|}=-i\frac{\eta_1}{\sqrt{2}}+\frac{\eta_2}{\sqrt{2}}+\mathcal{O}(\tau^{-1}).$$

We now choose other euclidean coordinates $\mathcal{F}$ by fixing the following orthonormal basis of $\mathbb{R}^3$:

$$f_2 = \frac{1}{|\xi'|}\begin{pmatrix} \xi^{(1)} \\ \xi^{(2)} \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3, \quad f_1 = f_2 \times f_3.$$

Here the vectors f_1, f_2, f_3 are expressed in the coordinates $\mathcal{E}$. In these new coordinates ξ, η_1 and η_2 read as

$$\xi = \begin{pmatrix} 0 \\ |\xi'| \\ \xi^{(3)} \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_2 = \frac{1}{|\xi|}\begin{pmatrix} 0 \\ -\xi^{(3)} \\ |\xi'| \end{pmatrix}.$$

Obviously, the metric e in these coordinates is still the identity matrix.

Therefore, ζ_1 and ζ_2 reads in these coordinates $\mathcal{F}$ as

$$\zeta_1 = \begin{pmatrix} i\left(\tau^2 + \frac{|\xi|^2}{4}\right)^{1/2} \\ -\frac{|\xi'|}{2} - (\tau^2 + k^2)^{1/2}\frac{\xi^{(3)}}{|\xi|} \\ -\frac{\xi^{(3)}}{2} + (\tau^2 + k^2)^{1/2}\frac{|\xi'|}{|\xi|} \end{pmatrix} \quad \zeta_2 = \begin{pmatrix} -i\left(\tau^2 + \frac{|\xi|^2}{4}\right)^{1/2} \\ \frac{|\xi'|}{2} - (\tau^2 + k^2)^{1/2}\frac{\xi^{(3)}}{|\xi|} \\ \frac{\xi^{(3)}}{2} + (\tau^2 + k^2)^{1/2}\frac{|\xi'|}{|\xi|} \end{pmatrix},$$

where $k^2 = \omega^2 \varepsilon_0 \mu_0$.

Consider $Z_1 = e^{i\zeta_1 \cdot x}(L_1 + R_1), Y_1$ the solutions stated in Proposition 3.2 corresponding to the pair $\tilde{\mu}_1, \tilde{\gamma}_1$ with $|\zeta_1| > C(\rho, M)$. Recall that

$$L_1 = \frac{1}{|\zeta_1|}\begin{pmatrix} \zeta_1 \cdot A_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2} B_1 \\ \hline \zeta_1 \cdot B_1 \\ \omega\varepsilon_0^{1/2}\mu_0^{1/2} A_1 \end{pmatrix}, \qquad \|R_1\|_{L^2(\Omega;\mathcal{Y})} \leq \frac{C(\rho, \Omega, M)}{|\zeta_1|}.$$

Additionally, consider $Y_2 = e^{i\zeta_2 \cdot x}(M_2 + S_2)$ the solutions stated in Proposition 3.3 corresponding to $\tilde{\mu}_2, \tilde{\gamma}_2$ with $|\zeta_2| > C(\rho, M)$. Also recall that

$$M_2 = \frac{1}{|\zeta_2|}\begin{pmatrix} \zeta_2 \cdot A_2 \\ -\zeta_2 \times A_2 \\ \hline \zeta_2 \cdot B_2 \\ \zeta_2 \times B_2 \end{pmatrix}, \qquad \|S_2\|_{L^2(\Omega;\mathcal{Y})} \leq \frac{C(\rho, \Omega, M)}{|\zeta_2|}.$$

Before plugging Z_1 and $\mathcal{Y}_2 = Y_2 - \dot{Y}_2$ into the estimate given in Proposition 4.1, we establish a quantitative version of the Riemann-Lebesgue lemma.

Lemma 4.4 Let τ be a positive parameter, $q \in L^1(\mathbb{R}^n)$ and

$$\omega_q(r) := \sup_{|y|<r} \|q - q(\cdot - y)\|_{L^1(\mathbb{R}^n)} .$$

Consider $\phi(\cdot;\tau) \in C^1(\mathbb{R}^n;\mathbb{R})$, then for any $0 < đ < 1$ one has

$$\left| \int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} q \, dV \right| \leq \omega_q(đ) + C đ^{-1} \sup_{x\in\mathbb{R}^n} \frac{1 + |\nabla\phi(x;\tau)|}{1 + |\nabla\phi(x;\tau)|^2} \|q\|_{L^1(\mathbb{R}^n)} .$$

Proof: Take $\varphi \in C_0^\infty(\mathbb{R}^n;\mathbb{R}_+)$ such that $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$ with

$$\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : |x| < 1\}$$

and denote $\varphi_đ = đ^{-n}\varphi(\cdot/đ)$. Then one has that

$$\int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} q \, dx = \int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} (q - \varphi_đ * q) \, dx + \int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} \varphi_đ * q \, dx.$$

On one hand,

$$\|q - \varphi_đ * q\|_{L^1(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \varphi(y) \|q - q(\cdot - đ y)\|_{L^1(\mathbb{R}^n)} \, dy.$$

On the other hand, since

$$(1 + \nabla\phi(\cdot;\tau)\cdot D) e^{i\phi(\cdot;\tau)} = (1 + |\nabla\phi(\cdot;\tau)|^2) e^{i\phi(\cdot;\tau)}$$

one has integrating by parts

$$\int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} \varphi_đ * q \, dx = \int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} \left(\frac{1 - \nabla\phi(\cdot;\tau)\cdot D}{1 + |\nabla\phi(\cdot;\tau)|^2} \right) (\varphi_đ * q) \, dx,$$

hence

$$\left| \int_{\mathbb{R}^n} e^{i\phi(\cdot;\tau)} \varphi_đ * q \, dx \right| \leq C đ^{-1} \sup_{x\in\mathbb{R}^n} \frac{1 + |\nabla\phi(x;\tau)|}{1 + |\nabla\phi(x;\tau)|^2} \|q\|_{L^1(\mathbb{R}^n)} .$$

$\square$

Recall that $\mu_j, \gamma_j \in H^{2+s}(\Omega)$ with $0 < s < 1/2$. In particular, $\partial^\alpha\mu_j, \partial^\alpha\gamma_j$ are in $H^s(\Omega)$ for $0 \leq |\alpha| \leq 2$. Moreover, the extension by zero allows to identify $H^t(\Omega) = H_0^t(\Omega)$ for $-1/2 < t < 1/2$ (see Proposition 2.2). Hence, $\mathbf{1}_\Omega(\partial^\alpha\mu_j)$ and $\mathbf{1}_\Omega(\partial^\alpha\gamma_j)$ are in $H_0^s(\Omega)$ which implies

$$\sup_{|y|<r} \|\mathbf{1}_\Omega\partial^\alpha\gamma_j - (\mathbf{1}_\Omega\partial^\alpha\gamma_j)(\cdot - y)\|_{L^1(\mathbb{R}^3)} \leq C r^s, \tag{4.14}$$

$$\sup_{|y|<r} \|\mathbf{1}_\Omega\partial^\alpha\mu_j - (\mathbf{1}_\Omega\partial^\alpha\mu_j)(\cdot - y)\|_{L^1(\mathbb{R}^3)} \leq C r^s. \tag{4.15}$$

This property can be found in [41].

Now we plugging Z_1 and $\mathcal{Y}_2 = Y_2 - \dot{Y}_2$ into the estimate given in Proposition 4.1 with different choices of A_j, B_j.

Choosing $B_1 = B_2 = 0$ and A_1, A_2 such that

$$i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot A_1 = i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot \overline{A_2} = 1$$

one gets, when τ becomes large, that

$$\begin{aligned}((Q_1 - Q_2)Z_1|Y_2)_\Omega &= \\ = \int_\Omega e^{-i\xi\cdot x}\left(\frac{1}{2}\Delta(\alpha_1-\alpha_2) + \frac{1}{4}(\nabla\alpha_1\cdot\nabla\alpha_1 - \nabla\alpha_2\cdot\nabla\alpha_2) + (\kappa_2^2 - \kappa_1^2)\right) dV \\ + \mathcal{O}((\tau^2 + |\xi|^2)^{-1/2}),\end{aligned}$$

where the implicit constant is $C(\rho, \Omega, M)$. In addition, by Lemma 4.4 with

$$\phi(x;\tau) = -|\xi'|x^2 + 2(\tau^2 + k^2)^{1/2}\frac{|\xi'|}{|\xi|}x^3$$

and (4.14), (4.15) one has

$$\left((Q_1 - Q_2)Z_1\middle|\dot{Y}_2\right)_\Omega = \mathcal{O}\left(\mathscr{d}^s + \frac{\mathscr{d}^{-1}|\xi|}{(|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2+k^2)|\xi'|^2)^{1/2}}\right).$$

Choosing $A_1 = A_2 = 0$ and B_1, B_2 such that

$$i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot B_1 = i\frac{\eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}} \cdot \overline{B_2} = 1$$

one gets, when τ becomes large, that

$$\begin{aligned}((Q_1 - Q_2)Z_1|Y_2) &= \\ = \int_\Omega e^{-i\xi\cdot x}\left(\frac{1}{2}\Delta(\beta_1-\beta_2) + \frac{1}{4}(\nabla\beta_1\cdot\nabla\beta_1 - \nabla\beta_2\cdot\nabla\beta_2) + (\kappa_2^2 - \kappa_1^2)\right) dV \\ + \mathcal{O}((\tau^2 + |\xi|^2)^{-1/2}),\end{aligned}$$

where the implicit constant is $C(\rho, \Omega, M)$. Again, by Lemma 4.4 and (4.14), (4.15) one has

$$\left((Q_1 - Q_2)Z_1\middle|\dot{Y}_2\right)_\Omega = \mathcal{O}\left(\mathscr{d}^s + \frac{\mathscr{d}^{-1}|\xi|}{(|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2+k^2)|\xi'|^2)^{1/2}}\right).$$

Denote

$$
\begin{aligned}
f &= \mathbf{1}_\Omega \left(\frac{1}{2}\Delta(\alpha_1 - \alpha_2) + \frac{1}{4}(\nabla\alpha_1 \cdot \nabla\alpha_1 - \nabla\alpha_2 \cdot \nabla\alpha_2) + (\kappa_2^2 - \kappa_1^2) \right) \\
g &= \mathbf{1}_\Omega \left(\frac{1}{2}\Delta(\beta_1 - \beta_2) + \frac{1}{4}(\nabla\beta_1 \cdot \nabla\beta_1 - \nabla\beta_2 \cdot \nabla\beta_2) + (\kappa_2^2 - \kappa_1^2) \right),
\end{aligned}
$$

where $\mathbf{1}_\Omega$ is the indicator function of Ω. By Proposition 4.1 and the properties of the special solutions, there exist three constants $c = c(\Omega)$, $C = C(\rho, \Omega, M)$ and $C' = C'(\rho, M)$ such that, for any $\tau \geq C'$ one has

$$
\begin{aligned}
|\widehat{f}(\xi)| + |\widehat{g}(\xi)| \leq C \Big(& B\big(\delta_C(C_\Gamma^1, C_\Gamma^2)\big) e^{c(\tau^2 + |\xi|^2)^{1/2}} + (\tau^2 + |\xi|^2)^{-1/2} \\
& + \text{\dj}^s + \frac{\text{\dj}^{-1}|\xi|}{(|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2 + k^2)|\xi'|^2)^{1/2}} \Big).
\end{aligned}
$$

Note that, for $R \geq 1$, one has

$$
\begin{aligned}
\|f\|^2_{H^{-1}(\mathbb{E})} + \|g\|^2_{H^{-1}(\mathbb{E})} &= \int_{|\xi|<R} (1 + |\xi|^2)^{-1} \big(|\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2 \big)\, d\xi \\
&\quad + \int_{|\xi|\geq R} (1 + |\xi|^2)^{-1} \big(|\widehat{f}(\xi)|^2 + |\widehat{g}(\xi)|^2 \big)\, d\xi \\
&\leq C \left(B\big(\delta_C(C_1, C_2)\big) e^{c(R+\tau)} + \tau^{-1} + \text{\dj}^s \right)^2 \int_0^R (1 + |r|^2)^{-1} r^2 \, dr \\
&\quad + C \int_{|\xi|<R} (1 + |\xi|^2)^{-1} \frac{\text{\dj}^{-2}|\xi|^2}{|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2 + k^2)|\xi'|^2}\, d\xi \\
&\quad + (1 + R^2)^{-1} \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right).
\end{aligned}
$$

Lemma 4.5 One has that

$$
\int_{|\xi|<R} (1 + |\xi|^2)^{-1} \frac{\text{\dj}^{-2}|\xi|^2}{|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2 + k^2)|\xi'|^2}\, d\xi \leq C \frac{R}{\text{\dj}^2 \tau}.
$$

Proof: Since $\{\xi \in \mathbb{R}^3 : |\xi| < R\} \subset \{\xi \in \mathbb{R}^3 : |\xi'| < R, |\xi^{(3)}| < R\}$, the integral in the statement is bounded by

$$
\int_{|\xi'|<R} \int_{|\xi^{(3)}|<R} (1 + |\xi|^2)^{-1} \frac{\text{\dj}^{-2}|\xi|^2}{|\xi|^2 + |\xi|^2|\xi'|^2 + 4(\tau^2 + k^2)|\xi'|^2}\, d\xi^{(3)}\, d\xi'.
$$

Changing to cylindrical coordinates it is enough to study

$$
I(R, \tau) := \int_{[0,R]\times[0,R]} (1 + r^2 + t^2)^{-1} \frac{r(r^2 + t^2)}{r^2 + t^2 + (r^2 + t^2)r^2 + 4(\tau^2 + k^2)r^2}\, dt\, dr.
$$

One has

$$
\begin{aligned}
I(R,\tau) \leq & \int_{[0,R]\times[0,R]} (1+r^2+t^2)^{-1} \frac{(r^2+t^2)^{1/2}}{(r^2+t^2+(r^2+t^2)r^2)^{1/2}} \\
& \times \frac{r(r^2+t^2)^{1/2}}{((r^2+t^2)r^2+4(\tau^2+k^2)r^2)^{1/2}}\,dt\,dr \\
= & \int_{[0,R]\times[0,R]} (1+r^2+t^2)^{-1} \frac{1}{(1+r^2)^{1/2}} \frac{r(r^2+t^2)^{1/2}}{((r^2+t^2)r^2+4(\tau^2+k^2)r^2)^{1/2}}\,dt\,dr \\
\leq & \sqrt{2}\int_{0<r<t<R} (1+r^2+t^2)^{-1} \frac{rt}{(t^2r^2+4(\tau^2+k^2)r^2)^{1/2}}\,dt\,dr \\
& + \sqrt{2}\int_{0<t<r<R} (1+r^2+t^2)^{-1} \frac{r^2}{(r^2r^2+4(\tau^2+k^2)r^2)^{1/2}}\,dt\,dr \\
= & 2\sqrt{2}\int_{0<r<t<R} (1+r^2+t^2)^{-1} \frac{t}{(t^2+4(\tau^2+k^2))^{1/2}}\,dt\,dr \\
= & 2\sqrt{2}\int_{[0,R]} \frac{t}{(t^2+4(\tau^2+k^2))^{1/2}} \frac{1}{(1+t^2)^{1/2}} \arctan\left(\frac{t}{(1+t^2)^{1/2}}\right) dt \leq C\frac{R}{\tau}.
\end{aligned}
$$

□

Therefore,

$$
\begin{aligned}
\|f\|_{H^{-1}(\mathbb{E})} + \|g\|_{H^{-1}(\mathbb{E})} \leq C\, \big(& B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big)e^{c(R+\tau)} + \tau^{-1}R^{1/2} \\
& + đ^s R^{1/2} + R^{1/2} đ^{-1}\tau^{-1/2} + R^{-1}\big).
\end{aligned}
$$

Now we choose R in such a way that $đ^s R^{1/2} + R^{1/2}đ^{-1}\tau^{-1/2}$ behaves as R^{-1}, that is,

$$
R = \frac{đ^{2/3}\tau^{1/3}}{(1+đ^{1+s}\tau^{1/2})^{2/3}},
$$

hence

$$
\|f\|_{H^{-1}(\mathbb{E})} + \|g\|_{H^{-1}(\mathbb{E})} \leq C\left(B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big)e^{c\tau} + \left(đ^s + \frac{1}{đ\tau^{1/2}}\right)^{2/3}\right).
$$

Choosing $\tau = đ^{-2(1+s)}$ the estimate becomes

$$
\|f\|_{H^{-1}(\mathbb{E})} + \|g\|_{H^{-1}(\mathbb{E})} \leq C\left(B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big)e^{cđ^{-2(1+s)}} + đ^{\frac{2s}{3}}\right).
$$

On the other hand, the a priori bound was chosen to have

$$
\|f\|_{H^s(\Omega)} + \|g\|_{H^s(\Omega)} \leq C(M),
$$

for $0 < s < 1/2$. Finally, the interpolation estimate (2.6) ensures the existence of two constants $C' = C'(\rho, M)$ and $C = C(\rho, \Omega, M, \omega)$ such that, for any $\not{d} \leq C'$, the following estimate holds

$$\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \leq C\left(B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big)e^{c\not{d}^{-2(1+s)}} + \not{d}^{\frac{2s}{3}}\right)^\theta \tag{4.16}$$

with $0 = -\theta + (1-\theta)s$.

The idea now is to transfer this estimate from f, g to the difference of the coefficients $\tilde{\mu}_1 - \tilde{\mu}_2$ and $\tilde{\gamma}_1 - \tilde{\gamma}_2$. This can be accomplished by using the Carleman estimate stated in Appendix B. A simple computation give:

$$\begin{aligned}
f &= \mathbf{1}_\Omega \tilde{\gamma}_1^{-1/2}\left[\Delta(\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2}) + q_f(\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2}) + p_f(\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2})\right],\\
g &= \mathbf{1}_\Omega \tilde{\mu}_1^{-1/2}\left[\Delta(\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2}) + q_g(\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2}) + p_g(\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2})\right];
\end{aligned}$$

where

$$\begin{aligned}
q_f &= -\left(\frac{\Delta\tilde{\gamma}_2^{1/2}}{\tilde{\gamma}_2^{1/2}} + \omega^2\tilde{\gamma}_1^{1/2}(\tilde{\gamma}_1^{1/2}\tilde{\mu}_1 + \tilde{\gamma}_2^{1/2}\tilde{\mu}_2)\right), \quad p_f = -\omega^2\tilde{\gamma}_1\tilde{\gamma}_2^{1/2}(\tilde{\mu}_1^{1/2} + \tilde{\mu}_2^{1/2}),\\
q_g &= -\left(\frac{\Delta\tilde{\mu}_2^{1/2}}{\tilde{\mu}_2^{1/2}} + \omega^2\tilde{\mu}_1^{1/2}(\tilde{\mu}_1^{1/2}\tilde{\gamma}_1 + \tilde{\mu}_2^{1/2}\tilde{\gamma}_2)\right), \quad p_g = -\omega^2\tilde{\mu}_1\tilde{\mu}_2^{1/2}(\tilde{\gamma}_1^{1/2} + \tilde{\gamma}_2^{1/2}).
\end{aligned}$$

Note that, thanks to the a priori bound, we have the following differential inequalities:

$$\begin{aligned}
|\Delta(\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2})| &\leq C(M)(|f| + |\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2}| + |\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2}|),\\
|\Delta(\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2})| &\leq C(M)(|g| + |\tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2}| + |\tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2}|).
\end{aligned}$$

In order to simplify the notation, we shall write $\phi_1 = \tilde{\gamma}_1^{1/2} - \tilde{\gamma}_2^{1/2}$ and $\phi_2 = \tilde{\mu}_1^{1/2} - \tilde{\mu}_2^{1/2}$. By the differential inequalities written above and the Carleman estimate, one has

$$\begin{aligned}
\sum_{j=1,2}\left(h\left\|e^{\varphi/h}\phi_j\right\|^2_{L^2(\Omega)} + h^3\left\|e^{\varphi/h}\nabla\phi_j\right\|^2_{L^2(\Omega;\mathbb{C}^3)}\right) \leq\\
\leq C''\sum_{j=1,2}\left(h^4\left\|e^{\varphi/h}\phi_j\right\|^2_{L^2(\Omega)} + h\left\|e^{\varphi/h}\phi_j\right\|^2_{L^2(\partial\Omega)} + h^3\left\|e^{\varphi/h}\nabla\phi_j\right\|^2_{L^2(\partial\Omega;\mathbb{C}^3)}\right)\\
+C''h^4\left(\left\|e^{\varphi/h}f\right\|^2_{L^2(\Omega)} + \left\|e^{\varphi/h}g\right\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

where the constant is $C'' = C''(\Omega, M)$ and $\varphi(x) = 1/2|x - x_0|^2$ with $x_0 \notin \overline{\Omega}$. The terms $h^4\left\|e^{\varphi/h}\phi_j\right\|^2_{L^2(\Omega)}$, with $j = 1, 2$, can be absorbed by the left hand side of the inequality. Hence, if $d_1 = \inf\{|x - x_0|^2 : x \in \Omega\}$ and $d_2 = \sup\{|x - x_0|^2 : x \in \Omega\}$

we get, for any $h < C''(\Omega, M)^{-1/3}$, that

$$e^{d_1/h} \sum_{j=1,2} \left(h \|\phi_j\|^2_{L^2(\Omega)} + h^3 \|\nabla \phi_j\|^2_{L^2(\Omega;\mathbb{C}^3)} \right) \leq C'' e^{d_2/h} \times$$
$$\left[h^4 \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right) + \sum_{j=1,2} \left(h \|\phi_j\|^2_{L^2(\partial\Omega)} + h^3 \|\nabla \phi_j\|^2_{L^2(\partial\Omega;\mathbb{C}^3)} \right) \right].$$

But now we can easily estimate

$$\begin{aligned}
\|\phi_1\|_{L^2(\partial\Omega)} + \|\nabla \phi_1\|_{L^2(\partial\Omega;\mathbb{C}^3)} &\leq C B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big),\\
\|\phi_2\|_{L^2(\partial\Omega)} + \|\nabla \phi_2\|_{L^2(\partial\Omega;\mathbb{C}^3)} &\leq C B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big),\\
\|\tilde\gamma_1 - \tilde\gamma_2\|_{L^2(\Omega)} + \|\nabla(\tilde\gamma_1 - \tilde\gamma_2)\|_{L^2(\Omega;\mathbb{C}^3)} &\leq C \left(\|\phi_1\|_{L^2(\Omega)} + \|\nabla \phi_1\|_{L^2(\Omega;\mathbb{C}^3)} \right),\\
\|\tilde\mu_1 - \tilde\mu_2\|_{L^2(\Omega)} + \|\nabla(\tilde\mu_1 - \tilde\mu_2)\|_{L^2(\Omega;\mathbb{C}^3)} &\leq C \left(\|\phi_2\|_{L^2(\Omega)} + \|\nabla \phi_2\|_{L^2(\Omega;\mathbb{C}^3)} \right).
\end{aligned}$$

The constants above depend on the a priori bounds M. With these inequalities and estimate (4.16), we obtain

$$\begin{aligned}
\|\tilde\gamma_1 - \tilde\gamma_2\|_{H^1(\Omega)} + \|\tilde\mu_1 - \tilde\mu_2\|_{H^1(\Omega)} &\leq C e^{\frac{d_2-d_1}{2h}} B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big)\\
&+ C e^{\frac{d_2-d_1}{2h}} \left(B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big) e^{c\cancel{d}^{-2(1+s)}} + \cancel{d}^{\frac{2s}{3}} \right)^{\frac{s}{1+s}},
\end{aligned}$$

where $d_2 > d_1$, $0 < s < 1/2$, $C = C(\rho, \Omega, M)$, $\cancel{d} \leq C'(\rho, M)$, $c = c(\Omega)$ and $h < C''(\Omega, M)^{-1/3}$. To end up with the estimate given in the statement, it is enough to choose the parameter $\cancel{d}$ as

$$\cancel{d}^{-2(1+s)} = -\frac{1}{2c} \log B\big(\delta_C(C^1_\Gamma, C^2_\Gamma)\big),$$

and to note that

$$0 < \frac{s^2}{3(s+1)^2} < \frac{s^2}{3}.$$

4.3 The domain U is partially spherical

Along this section we assume U to be a suitable partially spherical domain and we follow the notation in Definition 4.1 and Definition 4.2. Furthermore, n and ν will denote the outward unit normal forms of U and Ω, respectively.

The basic idea in this section is to use the Kelvin transform $\mathcal{K}$ to generalize our result on partially flat domain to the case of partially spherical domain. To achieve this, we study the behavior of Maxwell's equations and the distance δ_C under $\mathcal{K}$.

Note that $\mathcal{K} = \mathcal{K}^{-1}$ and $\mathcal{K}$ is a conformal transformation from (Ω, e) onto (U, e):

$$\mathcal{K}^* e = \frac{r_1^4}{|\boldsymbol{\cdot}|^4} e,$$

where $\mathcal{K}^*$ denotes the pull-back of $\mathcal{K}$.

Let $\tilde{E} = \mathcal{K}^* E$, $\tilde{H} = \mathcal{K}^* H$, $\tilde{\mu} = \mathcal{K}^* \mu$, and $\tilde{\gamma} = \mathcal{K}^* \gamma$. The following is the transformation law for Maxwell's equations under the Kelvin transform.

Lemma 4.6 One has $E, H \in H(U; \mathrm{curl})$ is solution of

$$dH + i\omega\gamma * E = 0 \qquad dE - i\omega\mu * H = 0$$

in U, if and only if, $\tilde{E}, \tilde{H} \in H(\Omega; \mathrm{curl})$ is a solution of

$$d\tilde{H} + i\omega\tilde{\gamma}\frac{r_1^2}{|\boldsymbol{\cdot}|^2} * \tilde{E} = 0 \qquad d\tilde{E} - i\omega\tilde{\mu}\frac{r_1^2}{|\boldsymbol{\cdot}|^2} * \tilde{H} = 0$$

in Ω.

Proof: The proof follows easily from

$$d\mathcal{K}^*\eta = \mathcal{K}^* d\eta, \quad \mathcal{K}^*(*\eta) = *_{\mathcal{K}^*e}\mathcal{K}^*\eta, \quad *_{ce}\eta = c^{3/2-k} * \eta.$$

Here η is k-form and c is an arbitrary positive smooth function. □

Lemma 4.7 Given $u_j \in H(U; \mathrm{curl})$ and $\tilde{v}_j \in H(\Omega; \mathrm{curl})$ with $j = 1, 2$, let us consider $\tilde{u}_j = \mathcal{K}^* u_j \in H(\Omega; \mathrm{curl})$ and $v_j = \mathcal{K}^* \tilde{v}_j \in H(U; \mathrm{curl})$.

(a) For any $z \in B^{1/2}(\partial U; \Lambda^1 T\mathbb{E})$ one has $\langle *(n \wedge u_j)|z\rangle = \langle *(\nu \wedge \tilde{u}_j)|w\rangle$, where $w = \mathcal{K}^* v|_{\partial\Omega} \in B^{1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$ with $v \in H^1(U; \Lambda^1 T\mathbb{E})$ such that $v|_{\partial U} = z$. Furthermore,

$$\|w\|_{B^{1/2}(\partial\Omega;\Lambda^1 T\mathbb{E})} \leq C \|z\|_{B^{1/2}(\partial U;\Lambda^1 T\mathbb{E})} . \tag{4.17}$$

For any $w \in B^{1/2}(\partial\Omega; \Lambda^1 T\mathbb{E})$ one has $\langle *(\nu \wedge \tilde{v}_j)|w\rangle = \langle *(n \wedge v_j)|z\rangle$, where $z = \mathcal{K}^* u|_{\partial U} \in B^{1/2}(\partial U; \Lambda^1 T\mathbb{E})$ with $u \in H^1(\Omega; \Lambda^1 T\mathbb{E})$ such that $u|_{\partial\Omega} = w$. Moreover,

$$\|z\|_{B^{1/2}(\partial U;\Lambda^1 T\mathbb{E})} \leq C \|w\|_{B^{1/2}(\partial\Omega;\Lambda^1 T\mathbb{E})} . \tag{4.18}$$

(b) For any $h \in B^{1/2}(\partial U)$ one has $\langle \mathrm{Div} *(n \wedge u_j)|h\rangle = \langle \mathrm{Div} *(\nu \wedge \tilde{u}_j)|g\rangle$ where $g = \mathcal{K}^* f|_{\partial\Omega} \in B^{1/2}(\partial\Omega)$ with $f \in H^1(U)$ such that $f|_{\partial U} = h$. Moreover,

$$\|g\|_{B^{1/2}(\partial\Omega)} \leq C \|h\|_{B^{1/2}(\partial U)} . \tag{4.19}$$

For any $g \in B^{1/2}(\partial\Omega)$ one has $\langle \mathrm{Div} *(\nu \wedge \tilde{v}_j)|g\rangle = \langle \mathrm{Div} *(n \wedge v_j)|h\rangle$ where $h = \mathcal{K}^* f|_{\partial U} \in B^{1/2}(\partial U)$ with $f \in H^1(\Omega)$ such that $f|_{\partial\Omega} = g$. Moreover,

$$\|h\|_{B^{1/2}(\partial U)} \leq C \|g\|_{B^{1/2}(\partial\Omega)} . \tag{4.20}$$

(c) The following estimates hold

$$\|*(n \wedge u_1) - *(n \wedge u_2)\|_{TH(\partial U)} \leq C \,\|*(\nu \wedge \tilde{u}_1) - *(\nu \wedge \tilde{u}_2)\|_{TH(\partial\Omega)}$$

and

$$\|*(\nu \wedge \tilde{v}_1) - *(\nu \wedge \tilde{v}_2)\|_{TH(\partial\Omega)} \leq C' \,\|*(n \wedge v_1) - *(n \wedge v_2)\|_{TH(\partial U)} .$$

Proof: The proof of the identities is an immediate consequence of the identities stated in the proof of Lemma 4.6 and the weak definitions of tangential trace and surface divergence. Proving (4.19), (4.20) is an easy computation and (4.17), (4.18) follow easily in coordinates from (4.19), (4.20) and (2.15). Finally, the estimates in (c) are a consequence of (a), (b), (2.9) and (2.10). □

Proposition 4.2 One has that

$$\delta_C(\tilde{C}^1_{\tilde{\Gamma}}, \tilde{C}^j_{\tilde{\Gamma}}) \leq C\delta_C(C^1_\Gamma, C^2_\Gamma),$$

where

$$C^j_\Gamma = C(\mu_j, \gamma_j; \Gamma), \quad \tilde{C}^j_{\tilde{\Gamma}} = C\left(\frac{r_1^2}{|\centerdot|^2}\tilde{\mu}_j, \frac{r_1^2}{|\centerdot|^2}\tilde{\gamma}_j; \tilde{\Gamma}\right),$$

with $j = 1, 2$.

Proof: Considering E_j, H_j and $\tilde{E}_j, \tilde{H}_j$ as u_j and $\tilde{v}_j$ in the statement of Lemma 4.7, this proposition is a consequence of Lemma 4.6 and the item (c) in Lemma 4.7. □

In order to end up with the proof in the case that U is partially spherical, it is enough to use Proposition 4.2 and recall that

$$\|\mu_1 - \mu_2\|_{H^1(U)} \leq C \left\| \frac{r_1^2}{|\centerdot|^2}(\tilde{\mu}_1 - \tilde{\mu}_2)\right\|_{H^1(\Omega)}$$
$$\|\gamma_1 - \gamma_2\|_{H^1(U)} \leq C \left\| \frac{r_1^2}{|\centerdot|^2}(\tilde{\gamma} - \tilde{\gamma}_2)\right\|_{H^1(\Omega)} .$$

Appendix A

Complements to the direct problem

In this appendix we state two results used to prove Theorem 1. The first one is due to Peetre and the second one is probably the main theorem in the analytic Fredholm theory.

A.1 Peetre's lemma

We state here the lemma in the form given and proven in the page 171 of [27].

Lemma A.1 (Peetre [36]) Let B_0, B_1, B_2 be three Banach spaces. Let $L_1 : B_0 \longrightarrow B_1$ and $L_2 : B_0 \longrightarrow B_2$ be two bounded linear maps satisfying:

(i) L_2 is compact.

(ii) There exists a constant $C > 0$ such that

$$\|u\|_{B_0} \leq C\left(\|L_1 u\|_{B_1} + \|L_2 u\|_{B_2}\right),$$

for any $u \in B_0$.

Then,

(a) $\operatorname{Ker} L_1$ has finite dimension and $\operatorname{Rang} L_1$ is closed,

(b) there exists a constant $C' > 0$ such that

$$\inf_{v \in \operatorname{Ker} L_1} \|u + v\|_{B_0} \leq C' \|L_1 u\|_{B_1}.$$

A.2 Analytic Fredholm theory

We state here the analytic Fredholm theorem in the form given and proven in [37]. Let H be a Hilbert space and let $\mathcal{L}(H)$ denote the space of bounded linear operators.

Theorem A.1 (analytic Fredholm theorem) Let D be and open connected subset of $\mathbb{C}$. Let $f : D \longrightarrow \mathcal{L}(H)$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then, either $(I - f(z))^{-1}$ exists for no $z \in D$, or $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$ where S is a discrete subset of D (i. e. a subset which has no limit points in D). In this case, $(I - f(z))^{-1}$ is meromorphic in D, analytic in $D \setminus S$, the residues at the poles are finite rank operators and if $z \in S$ then $f(z)u = u$ has a nonzero solution in H.

Appendix B

A Carleman estimate

Here we give the Carleman estimate used in the proof of Theorem 2 and Theorem 4. A proof of this estimate can be found in [18].

Proposition B.1 Let φ be defined by $\varphi(x) = 1/2|x - x_0|^2$ with $x_0 \notin \overline{\Omega}$. There exists a positive constant C such that, for all $h \leq 1$ and any function ϕ smooth enough, the following estimate holds

$$h \left\|e^{\varphi/h}\phi\right\|^2_{L^2(\Omega)} + h^3 \left\|e^{\varphi/h}\nabla\phi\right\|^2_{L^2(\Omega;\mathbb{C}^3)} \leq$$
$$\leq C \left(h^4 \left\|e^{\varphi/h}\Delta\phi\right\|^2_{L^2(\Omega)} + h \left\|e^{\varphi/h}\phi\right\|^2_{L^2(\partial\Omega)} + h^3 \left\|e^{\varphi/h}\nabla\phi\right\|^2_{L^2(\partial\Omega;\mathbb{C}^3)}\right).$$

The constant here depends on the distance from x_0 to Ω and on the diameter of Ω.

Appendix C

Proof of Lemma 1.1

In this third part of the appendix, we give a proof of Lemma 1.1.

Consider $\sigma \in C^{1,1}(\mathbb{R}^3_-) \cap C^1(\overline{\mathbb{R}^3_-})$. Define γ as $\gamma(x) = \sigma(x)$ for $x \in \mathbb{R}^3_-$ and $\gamma(x) = \sigma(\mathcal{R}(x))$ for $x \in \mathbb{R}^3_+$, where $\mathcal{R}(x^1, x^2, x^3) = (x^1, x^2, -x^3)$. Note that $\gamma \in C^{0,1}(\mathbb{R}^3)$.

Lemma C.1 Let $v \in H^1(\mathbb{R}^3)$ be such that $-\Delta v + qv \in L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \gamma^{1/2}(-\Delta v + qv)\varphi \, dx - 2\int_{\{x^3=0\}} v\partial_{x^3}\sigma^{1/2}\varphi \, dx' = 0 \tag{C.1}$$

for all $\varphi \in H^1(\mathbb{R}^3)$ with $q(x) = (\sigma^{-1/2}\Delta\sigma^{1/2})(x)$ for $x \in \mathbb{R}^3_-$ and $q(x) = (\sigma^{-1/2}\Delta\sigma^{1/2})(\mathcal{R}(x))$ for $x \in \mathbb{R}^3_+$. Then $u = \gamma^{-1/2}v \in H^1(\mathbb{R}^3)$ is a weak solution of $\nabla \cdot (\gamma\nabla u) = 0$ in $\mathbb{R}^3$.

Proof: Let $\gamma_\pm$ denote $\gamma|_{\mathbb{R}^3_\pm}$. Plugging arbitrary $\varphi_+ \in C_0^\infty(\mathbb{R}^3_+)$ and arbitrary $\varphi_- \in C_0^\infty(\mathbb{R}^3_-)$ one gets that

$$\begin{aligned} \gamma_+^{1/2}(-\Delta v + qv) = 0 \quad \text{in} \quad \mathbb{R}^3_+, \\ \gamma_-^{1/2}(-\Delta v + qv) = 0 \quad \text{in} \quad \mathbb{R}^3_-. \end{aligned} \tag{C.2}$$

One can define $-\gamma_+^{1/2}\partial_{x^3}v|_{\{x^3=0\}}$ and $\gamma_-^{1/2}\partial_{x^3}v|_{\{x^3=0\}}$ in a weak sense:

$$\left\langle -\gamma_+^{1/2}\partial_{x^3}v \middle| f \right\rangle = \int_{\mathbb{R}^3_+} \nabla v \cdot \nabla(\gamma_+^{1/2}\psi_+) \, dx + \int_{\mathbb{R}^3_+} qv\gamma_+^{1/2}\psi_+ \, dx$$

and

$$\left\langle \gamma_-^{1/2}\partial_{x^3}v \middle| f \right\rangle = \int_{\mathbb{R}^3_-} \nabla v \cdot \nabla(\gamma_-^{1/2}\psi_-) \, dx + \int_{\mathbb{R}^3_-} qv\gamma_-^{1/2}\psi_- \, dx$$

where $\psi_\pm \in H^1(\mathbb{R}^3_\pm)$ such that $\psi_\pm|_{\{x^3=0\}} = f$. Note that a priori $-\gamma_+^{1/2}\partial_{x^3}v|_{\{x^3=0\}}$ is not the oppose of $\gamma_-^{1/2}\partial_{x^3}v|_{\{x^3=0\}}$.

By hypothesis $-\Delta v + qv \in L^2(\mathbb{R}^3)$, then (C.2) implies that

$$\gamma^{1/2}(-\Delta v + qv) = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Hence

$$\begin{aligned}\left\langle -\gamma_+^{1/2}\partial_{x^3} v \middle| f \right\rangle + \left\langle \gamma_-^{1/2}\partial_{x^3} v \middle| f \right\rangle &= \int_{\mathbb{R}^3} \nabla v \cdot \nabla(\gamma^{1/2}\psi)\, dx + \int_{\mathbb{R}^3} qv\gamma^{1/2}\psi\, dx \\ &= \int_{\mathbb{R}^3} \gamma^{1/2}(-\Delta v + qv)\psi\, dx = 0\end{aligned}$$

with $\psi \in H^1(\mathbb{R}^3)$ such that $\psi|_{\{x^3=0\}} = f$.

Again by (C.1) one gets that

$$\partial_{x^3}\sigma^{1/2} v|_{\{x^3=0\}}.$$

Consider $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3} \gamma\nabla(\gamma^{-1/2}v)\cdot\nabla\varphi\, dx &= \int_{\mathbb{R}^3} (-\nabla\gamma^{1/2}v + \gamma^{1/2}\nabla v)\cdot\nabla\varphi\, dx \\ = \int_{\mathbb{R}^3_+} (-\nabla\gamma_+^{1/2}v + \gamma_+^{1/2}\nabla v)\cdot\nabla\varphi\, dx &+ \int_{\mathbb{R}^3_-} (-\nabla\gamma_-^{1/2}v + \gamma_-^{1/2}\nabla v)\cdot\nabla\varphi\, dx \\ = \int_{\mathbb{R}^3_+} (\Delta\gamma_+^{1/2}v - \gamma_+^{1/2}\Delta v)\cdot\varphi\, dx &+ \left\langle -\gamma_+^{1/2}\partial_{x^3} v \middle| \varphi|_{\{x^3=0\}} \right\rangle \\ + \int_{\mathbb{R}^3_-} (\Delta\gamma_-^{1/2}v - \gamma_-^{1/2}\Delta v)\cdot\varphi\, dx &+ \left\langle \gamma_-^{1/2}\partial_{x^3} v \middle| \varphi|_{\{x^3=0\}} \right\rangle = 0.\end{aligned}$$

□

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